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Attack and Interception in Networks

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Abstract

This paper studies a game of attack and interception in a network, where a single attacker chooses a target and a path, and each node chooses a level of protection. We show that the Nash equilibrium of the game exists and is unique. It involves a mixed strategy of the attacker except when one target has a very high value relative to others. We characterize equilibrium attack paths and attack distributions as a function of the underlying network and target values. We also show that adding a link or increasing the value of a target may harm the attacker - a comparative statics effect which is reminiscent of Braess's paradox in transportation economics. Finally, we contrast the Nash equilibrium with the equilibria of two variations of the model: one where nodes make sequential protection decisions upon observing the arrival of a suspicious object, and one where all nodes cooperate in defense.

Keywords: Network interdiction, Networks, Attack and defense, Inspection
Declarations of interest: none
JEL: D85, C72, K42

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1 Introduction

Networks are often used to transport troops, bombs, drugs, contraband goods, or viruses. Preventing or stopping the transportation of illegal and dangerous objects on networks has long been the goal of army and police forces, customs and cybersecurity agents. In this paper, we analyze a game between an attacker who chooses a target node in the network and nodes who build up defense to deter the attack.

This issue is connected to the vast literature in operations research on *network interdiction*. Models of network interdiction involve two players: the *interdictor* who changes the structure of the network (for example, placing detection devices, destroying links or limiting capacities on links) and the *evader* who uses the network to transport objects from a source to a sink. Typically, the game played has a Stackelberg structure: the interdictor moves first and the evader second. The objectives of the two players are diametrically opposed: the evader wants to maximize the flow of goods or minimize the length of the path between the source and the sink whereas the interdictor wants to minimize the evader's objective.

The problem that we focus on in this paper is different in two important respects. First, a fundamental assumption of models of network interdiction is that the source and sink are exogenously given. While this assumption is easily justified in some contexts - for instance troop movements or delivery of goods to specific locations, it is less likely to be justified for terrorist organizations, drug smugglers or hackers. They typically choose endogenously their target destination(s), at least partly to make it more difficult for their opponent(s) to prevent successful attacks. Second, in our benchmark model, we consider a decentralized structure where each node only cares about the possible damage to itself. Hence, there is no coordination in decisions about whether to defend a specific node. In particular, nodes that believe that they are not likely to be attacked will not invest in defense.

We model a $(n+1)$- person game between an attacker A and n target nodes. Each target has a different value for the attacker. Each target node i is a distinct *player*, and can invest in a technology to intercept the object with some probability at a quadratic cost. The attacker is interpreted as a terrorist or criminal, whose objective is to transport an object (a bomb, biological agent, contraband goods, or a packet of drugs) from his location (labeled 0) to one of the defenders' locations. The defenders are interpreted as communities, cities or airports that can be targets of terrorist attacks or markets for drugs or contraband goods. The defenders are connected by an undirected network $\mathcal{G}$ which is interpreted as a transportation network (e.g. airline or road network) that can be used by the attacker to transport the object. The network also connects the attacker to the defenders. We take the network to be exogenously given in the analysis. In our baseline model, we assume that the detection technology involves a fixed cost. All players choose their strategies simultaneously. Each node has to take into account the probability that it is under attack as well as the decisions on defense of other nodes. For instance, if a network is a tree and a node believes that there is a good chance that the attacker will be intercepted by one of the predecessor nodes, then it is likely to spend less on its defense. Similarly, the attacker has to take into account the defense plans of the nodes in order to maximize chances of a successful attack. The appropriate solution concept for this situation of strategic interaction is Nash equilibrium.

Our main result shows that there is a unique Nash equilibrium in the baseline model. We identify conditions under which this unique equilibrium will be in pure strategies. Essentially, a pure strategy equilibrium can exist if and only if one target node has sufficiently high value so that it is worthwhile to attack even if the node is heavily defended. Otherwise, there will be a mixed strategy equilibrium. Our proof of uniqueness also casts some light on the equilibrium strategies. We show that the attacker will use only *one* path to attack a node in the support even if there are possibly multiple paths of attack. The equilibrium attack tree can easily be characterized, using the fact that if a node is preceded by several targets, it will always be attacked through the preceding target with lowest value. Equilibrium attack probabilities are characterized as the solution to a simple system of equations.

We then go on to describe *two* kinds of comparative statics results. In one, we discuss how the equilibrium outcomes change when an additional link is added to the network. In the second type, we describe how the equilibrium outcomes change when there is a *small* increase in the value of a target node. It turns out that in both cases, the changes in equilibrium payoff to the attacker can be counterintuitive. In particular, an additional link should *increase* the payoff of the attacker since this gives the attacker more options of attack. Intuition should suggest that since the attacker can always choose not to use the new link, it can never be worse off in the network after the new link is available. However, we show that this type of revealed preference argument is not valid in our model. We show that an analogue of *Braess's Paradox* can occur in our model - the new link *must* be used in equilibrium and the attacker's equilibrium payoff can be strictly lower! A similar counterintuitive result can occur even in the second case - the attacker's equilibrium payoff may be strictly lower when the value of a node increases.

We go on to study a variation of the model, where nodes do not commit ex ante to their interception strategy but instead choose how much to spend on defense upon observing a suspicious package. In this version of the model, moves are sequential and we compute the sequential equilibria of the extensive form game. We first show that under relatively weak genericity assumptions, there cannot be a mixed strategy sequential equilibrium when $\mathcal{G}$ is a tree. We characterize the set of pure strategy targets in lines, and show that equilibrium is no longer unique: the attacker can choose different targets, giving rise to different beliefs off the equilibrium path.

Finally, we look at an alternative formulation where a single *centralized* agency coordinates defense. In this case, *all* nodes - even nodes that are not targets - on an attack path are defended. In contrast to the baseline model, the attacker may use multiple paths of attack to the same target in a general network. We focus on the line and show that the equilibrium is unique. We also identify conditions under which all nodes in the line will be attacked in equilibrium and compare the equilibrium outcomes under the cooperative and non-cooperative formulations. Not surprisingly, the attacker is worse off in the case of centralized defense. We show that all nodes but the first target are better off in the case of centralized defense, but that the first target may either be better off or worse off. Hence, the first target may need to be subsidized to accept to participate in a centralized defense scheme.

All proofs are given in an appendix.

2 Related literature

Our paper is related to two strands of the literature: the operations research literature on network interdiction and the economics literature on attack and defense in networks. The extensive literature on network interdiction originated with Wollmer [20]'s characterization of the arc to be removed to minimize the flow between a source and a sink in the network. Three types of problems have been considered. In shortest path interdiction, (Golden [13],) the objective of the evader is to minimize the length of the path between the source and the sink. In most reliable path interdiction, (Washburn and Wood [19]), the interdictor places detection devices on the edges, and the objective of the evader is to minimize the probability of detection. In network flow interdiction, (McMasters anbd Mustin [16], Ghare, Montgomery and Turner [11], and Fulkerson, Ray and Harding [10]), edges are capacitated and the objective of the evader is to maximize the flow between the source and the sink. The applications range from the disruption of enemy troop movement (McMasters and Mustin [16] and Ghare, Montgomery and Turner [11]) to drug smuggling (Wood [21] and Washburn and Wood [19]) and the detection of nuclear material (Morton, Pan and Saeger [17]). The literature is very clearly summarized in Collado and Papp [4] and surveyed in Smith and Song [18].

Our model is closest to the detection game studied in Washburn and Wood [19] where the interdictor and evader play a simultaneous game. The interdictor chooses detection devices on edges and the evader chooses a mixed strategy over paths. However, Washburn and Wood [19] assumes that the source and sink are fixed. In the stochastic network interdiction problem (Cormican, Morton and Wood [5]), this assumption is relaxed and source and sink are drawn from a stochastic distribution. However, the distribution is exogenous and cannot be controlled by the evader. Finally, a major point of departure between the operations research literature and our paper stems from the questions asked and methods used. The literature on network interdiction focuses attention on the complexity of the integer and linear programming problems involved in network interdiction and studies algorithms to find exact or approximate solutions. Instead, we study attack and interception as a game, provide an exact characterization of equilibrium and compute the comparative statics effects of changes in the parameters of the problem.

Economists, using formal game theoretic models have also contributed to the literature on conflict in networks. Some of these papers focus on network design and emphasize tradeoffs between connectivity and ease of external disruptions. The literature started with Dziubinski and Goyal [7], who model attacks on infrastructure, focusing on the trade off between cost of network defense and connectivity. Dziubinski and Goyal [8] is also a model of attack and defense in infrastructure networks. The defender chooses to protect "nodes" of a given network against attacks at some cost. A defended node is immune to attack while an undefended node and all its links is removed if attacked. The focus is thus on the key nodes that need to be defended in order to ensure efficient functionality. Goyal and Vigier [14] models hacking and cybersecurity. The defender moves first, constructs the network and chooses an allocation of defense resources to defend nodes. The attacker then chooses an attack strategy, and how to spread through the network by using successful resources. The paper uses a Tullock contest function to model the outcome between the defender and attacker at any node. Successful attacks travel from node to node in the network, the"contagion" representing the spread of computer viruses. Cerdeiro, Dziubinski and Goyal [3] considers a version of the model where defense is decentralized and the game is played, as in our model, between a single attacker and defenders located at each node of the network. Finally, Bloch, Dutta and Dziubinski [1] study network design in a different game of attack and defense, where the objective of the defender is to hide an object inside the network. While our paper considers, like the earlier literature, a game played between an attacker and defender(s) on a network, our model is very different from existing models. At least two differences are worth emphasizing. First, we do not express payoffs as a function of the nodes captured and the residual network as in previous models. Second, we explicitly take into account paths of attack as objects may be intercepted before reaching their target - a feature which is absent from the existing games of attack and defense in networks.

3 The Game

In our baseline model, we consider the following $(n+1)$-person game between an attacker (player 0) and n defenders. There is an exogenously given graph $\mathcal{G}$ on the $(n+1)$ nodes. A *path* originating at 0 in the network, p, is a sequence of distinct points $i_1, .., i_m$ such that $g(0, i_1) = g(i_1, i_2) = g(i_2, i_3) = .. = g(i_{m-1} i_m) = 1$. A path thus describes a sequence of moves along the transportation network, where none of the locations are visited twice.

Each defender has a *value* b_i, which is normalized to be smaller than 1. The value b_i captures the strategic or symbolic importance of a terrorist target or the profitability of the market for drugs or contraband goods.

We assume that the defenders invest in a technology to intercept the object. We let x_i denote the probability that defender i detects and intercepts the object, and assume that the technology allowing defender i to detect with probability x_i has a quadratic cost $c(x_i) = \frac{1}{2} x_i^2$.[1] In the baseline model, each defender chooses its detection probability independently.

The attacker chooses to attack a target i. In principle, there are several paths between 0 and the target i, and the attacker can use a mixed strategy by randomising between different paths of attack. However, we will soon show that the

[1]Alternatively, we could assume that each defender makes a binary decision to invest in a fixed detection technology and that the fixed cost of the technology is random and drawn from a uniform distribution over $[0, 1]$. Both models give rise to the same detection probability x_i which in the latter model is interpreted as the threshold value of the fixed cost for which the defender chooses to invest in the detection technology.

attacker will choose a specific path to attack any given target. Of course, the attacker can *mix* between different targets. If the attacker reaches the target i and is undetected, he receives a payoff of b_i and defender i suffers a cost $-b_i$. We assume that there are no spillover effects, so that all other targets j are unaffected by the fact that the attacker has successfully reached his target i.

We provide an informal description of the game. The attacker's (pure) strategy is to choose a target i as well as the path(s) of attack from 0 to i. Each defender j' s strategy is to choose the probability x_j of intercepting the object. All strategies are chosen simultaneously. The payoff of the attacker if his attack on i is successful is given by b_i, and is 0 if his attack is foiled. The payoff of defender i if he is successfully attacked is given by $-b_i - \frac{1}{2}x_i^2$ and the payoff of defender j who is not attacked is given by $-\frac{1}{2}x_j^2$. Observe that the game is *not* a zero-sum game because the defenders incur an additional cost of investing in the detection technology.

Given any path p and $i \in p$, the set of *predecessors* of i in p is $P(p,i) = \{j \in p | j \text{ lies on the path between 0 and } i\}$. Fix a defense vector $x = (x_1, \ldots, x_n)$. For any node i contained in the path p, we let $\alpha_i(p)$ denote the probability that the object sent along path p reaches node i:

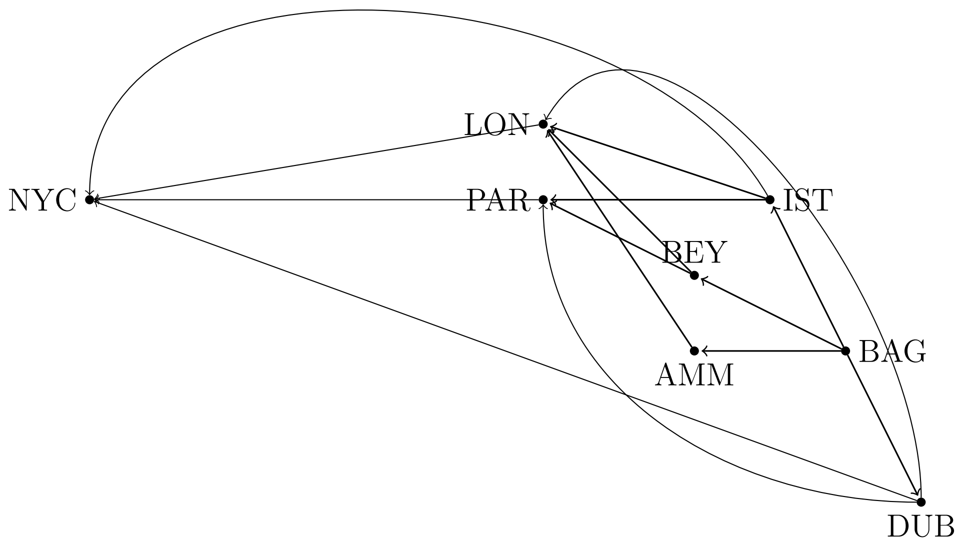
$$\alpha_i(p) = \prod_{j \in P(p,i)} (1 - x_j).$$

so that the probability that the good reaches node i and is not detected at i is given by

$$\beta_i(p) = \alpha_i(p)(1 - x_i).$$

The expression $\beta_i(p)$ thus expresses the notion that if the attacker wants to successfully attack target i along any given path p, then the object has to pass undetected at all nodes preceding i on the path p.

To illustrate the model, consider a (hypothetical) situation where a terrorist located in Baghdad plans to attack a target in Europe or the United States. Based on data on airline routes from flightsfrom.com in December 2020, the following network describes the shortest routes from Baghdad to London, New York and Paris.



We suppose that the cities can be partitioned into three groups according to target values: cities in the Middle-East, cities in Europe and New York City. The ranking within each group puts Dubai above Istanbul, Istanbul above Beirut, Beirut above Amman and London above Paris. In order to capture this ranking, we assign the following target values:

Amman	0.6
Beirut	0.61
Dubai	0.63
Istanbul	0.62
London	0.71
New York	0.8
Paris	0.7

The objective of our analysis is to study which targets and paths are chosen by the terrorist, and how the different cities respond by investing in defense technologies.

4 Equilibrium Analysis

In the benchmark model, we have assumed that the attacker and the defenders move simultaneously. A Nash equilibrium is thus a probability distribution over targets specifying as well the paths of attack, and a vector of investments $(x_1, ..., x_n)$ such that (i) the attacker chooses his optimal attack strategy given the vector of detection probabilities $(x_1, ..., x_n)$ and (ii) each defender i chooses her optimal investment strategy given the investment strategies of the other defenders x_{-i} and the attacker's strategy.

Unless otherwise stated, we use the following generic Assumption throughout the paper.

Assumption 1. *For any two defenders i, j, $b_i \neq b_j$.*

We start with a few preliminary Lemmas describing the best responses of the attacker.

For any mixed strategy of the attacker, we use Δ to denote the set of targets that are attacked with positive probability.[2] Fix also any vector of defense strategies $x = (x_1, \ldots, x_n)$ satisfying the restriction that $x_i = 0$ if $i \notin \Delta$.

On any path p, for any $i, j \in p \cap \Delta$, we will say that j is an *immediate* predecessor of i in p given π if $j \in P(p, i)$ and $(P(p, i) \setminus \{j\}) \cap \Delta \subset P(p, j) \cap \Delta$. That is, j is an immediate predecessor of i if j is a predecessor of i and there are no other predecessors of i in $p \cap \Delta$.

Lemma 1. *Fix the defense investments x and consider any best response of the attacker to x.*

(i) If j is an immediate predecessor of i in p, then

$$b_i(1 - x_i) = b_j.$$

(ii) If $i, j \in \Delta$ and there exist attack paths p, p' such that i and j are the first targets in paths p and p', then

$$b_i(1 - x_i) = b_j(1 - x_j).$$

Lemma 1 is a direct consequence of the fact that the attacker must be indifferent between any target in the support Δ. We use this Lemma to derive another important result on the equilibrium strategy of the attacker.

Lemma 2. *For any two paths p, p', if i is attacked along paths p and p', then $P(p, i) \cap \Delta = P(p', i) \cap \Delta$.*

Lemma 2 shows that, without loss of generality, we can assume that any target i is attacked from a single path p in equilibrium. If the attacker uses two paths p and p' to reach a target i, the two paths only differ on nodes which are not attacked, and hence never invest in the detection technology. Hence the probability that the attack is successful is identical along the two paths, $\beta_i(p) = \beta_i(p')$. Therefore, we can simplify notation considerably by identifying targets with the *single* path of attack. So, we can characterize an equilibrium strategy of the attacker with the probability distribution q over targets in Δ.

Moreover, since every target is attacked through only one path, the attacker is connected to all targets in Δ though a *tree* $\mathcal{T} \subseteq \mathcal{G}$ rooted at 0. We also note

[2] By adopting this notation, we do not differentiate between different paths of attack to the same target. We show below that this omission is without loss of generality since the attacker will choose a *unique* attack path for each target.

that we can now define predecessors of targets without reference to paths since effectively there is only one path to each target. So, $P(i)$ will denote the set of predecessors in $\mathcal{T}$. For any target $i \in \Delta$ we let $\delta(i)$ denote the number of targets in Δ along the unique path connecting 0 to i in $\mathcal{T}$. The integer $\delta(i)$ is interpreted as the distance between the attacker's node and target i. If a node i is the first node attacked on an attack path, it has distance $\delta(i) = 0$. By convention, we let Δ_m be the set of all targets at distance m from the attacker:

$$\Delta_m = \{i \in \Delta | \delta(i) = m\}.$$

Using the simpler notation, let us formally define the expected payoffs of the attacker and defenders for any $(n+1)$ tuple of strategies $(q, x_1, \ldots, x_n)$.

$$U_0(q, x_1, \ldots, x_n) = \sum_{i=1}^{n} \beta_i q_i b_i \tag{1}$$

$$U_i(q, x_1, \ldots, x_n)) = -\alpha_i(1 - x_i) q_i b_i - \frac{x_i^2}{2} \tag{2}$$

For later reference, we also derive the best response of any defender i to (q, x_{-i}). This is obtained from the first order condition for a maximum of equation 2, and is

$$x_i(q, x_{-i}) = \alpha_i q_i b_i \tag{3}$$

We first note that a Nash equilibrium of the game of attack and interception always exists by appealing to the Debreu-Fan-Glicksberg fixed point theorem.

Theorem 1. *The game of attack and interception on a network always admits a Nash equilibrium in mixed strategies.*

Our second result characterizes situations under which the equilibrium of the game is in pure strategies.

Proposition 1. *The game admits an equilibrium in pure strategies if and only if there exists a defender i such that*

(i) $b_i(1 - b_i) \geq b_j$ *for all j such that there is a path p, $i \notin P(p, j)$.*

(ii) $b_i \geq b_j$ *for all j such that for all paths p, $i \in P(p, j)$.*

Proposition 1 shows that an equilibrium in pure strategies only occurs when one of the targets has a value which is much larger than the value of any other target. It is only in these very asymmetric situations that the attacker has an incentive to concentrate his attack on one of the nodes which in turn will choose the highest level of defense, while all other nodes remain unprotected.

We next show that the game admits a *unique* Nash equilibrium. A series of Lemmas precedes the main result.

Lemma 3. *Consider any equilibrium (q, x^*), with $\Delta = \{i \in N : q_i > 0\}$. Then,*

1. *Suppose that $i \in \Delta$. Then, if $j \in \Delta \cap P(i)$, $b_j < b_i$.*

2. *Suppose that $i \in \Delta$, and there exists a path between i and j in $\mathcal{G}$ which does not intersect Δ. Then if $b_j > b_i$, we must have $j \in \Delta$.*

Lemma 3 first shows that along an equilibrium path, the values of targets must be increasing, and that any defender which is preceded by another defender with higher value cannot be attacked. However, Lemma 3 does not pin down the path by which a target is attacked. The next Lemma helps us identify this path for a particular target i.

Lemma 4. *Suppose that $i \in \Delta_m$ with $m \geq 1$. Let $k \in \Delta$ be an immediate predecessor of i on the unique path from 0 to i in $\mathcal{T}$. Let*

$$J = \{j \in \Delta \text{ there exists a path from } j \text{ to } i \text{ in } \mathcal{G} \text{ which does not intersect } \Delta\}$$

Then $b_k < b_j$ for all $j \in J$.

Lemma 4 shows that target i will be attacked from the target k with the *lowest* value among all the targets for which there exists a path which does not intersect Δ. This Lemma is used to characterize the equilibrium attack paths whenever the support Δ is given.

Furthermore, once the attack paths are given, the equilibrium attack probabilities q_i, defense investments x_i and equilibrium utility of the attacker U can be computed as solutions to the following system of equations:

$$
\begin{aligned}
x_i &= 1 - \frac{U}{b_i} \text{ if } i \in \Delta_0, && (4)\\
q_i &= \frac{1}{b_i}(1 - \frac{U}{b_i}) \text{ if } i \in \Delta_0, && (5)\\
x_i &= 1 - \frac{b_{k(i)}}{b_i} \text{ if } i \notin \Delta_0, && (6)\\
q_i &= \frac{b_{k(i)}}{b_i U}(1 - \frac{b_{k(i)}}{b_i}) \text{ if } i \notin \Delta_0. && (7)\\
\sum_i q_i &= 1, && (8)
\end{aligned}
$$

where $k(i)$ denotes the immediate predecessor of i along the attack path.

To understand these formulas, note that the equilibrium investment levels are obtained from the equations guaranteeing that the attacker is indifferent among the targets in the support, so that

$$
\begin{aligned}
b_i(1 - x_i) &= U \text{ for } i \in \Delta_0,\\
b_i(1 - x_i) &= b_{k(i)} \text{ for } i \notin \Delta_0
\end{aligned}
$$

providing two different expressions, whether i is the first target along an attack path or not. The optimal choice of defense investments must satisfy equation (3) which we rewrite as

$$x_i = q_i \frac{U}{1 - x_i}.$$

so that

$$q_i = \frac{x_i(1 - x_i)}{U}.$$

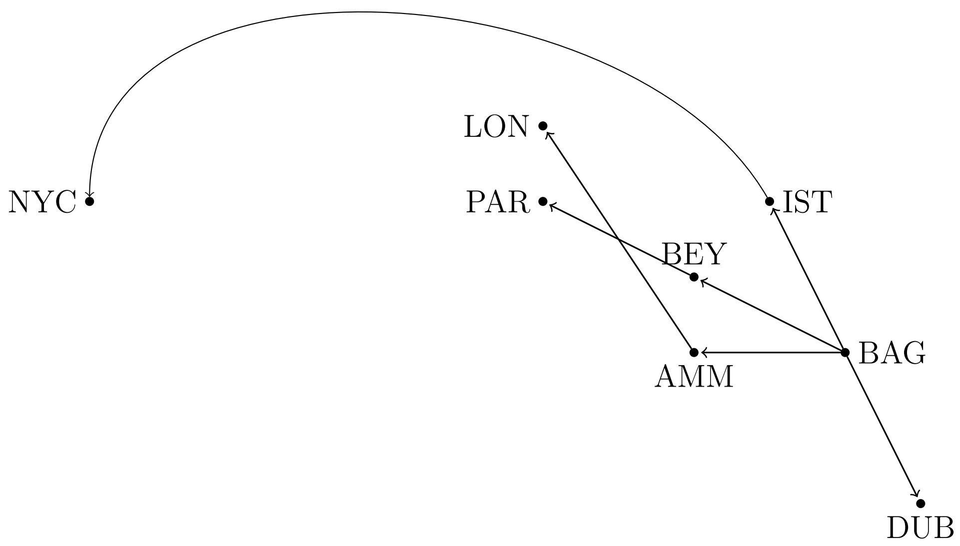
Replacing x_i with the values in equations (4) and (6) gives the expressions for the equilibrium attack probabilities.

Because all probabilities q_i are strictly decreasing functions of U, equation (8) has a unique solution, establishing that the equilibrium probabilities are unique. Hence, once the support Δ is given, the equilibrium attack paths and attack probabilities are easily characterized.

Theorem 2 shows that there is a unique equilibrium support Δ, and hence a unique equilibrium. The proof relies on the following arguments. Suppose that there were two different equilibrium supports Δ and Δ'. As one node is attacked under Δ but not under Δ', the equilibrium utility of the attacker must be at least as large under Δ than Δ'. But this implies that the support Δ' must be strictly contained in the support Δ. Next, we show that the same attack paths must be used in the two equilibria, which, together with the fact that the equilibrium utility is at least as large in the equilibrium with support Δ', imply that the defense investments and the equilibrium probabilities are lower under Δ' than Δ. This final observation leads to a contradiction, as the sum of probabilities is equal to one in both equilibria, and the support Δ' is strictly contained in Δ.

Theorem 2. *Given Assumption 1, the game of attack and interception admits a unique Nash equilibrium.*

We illustrate the unique Nash equilibrium of the game in the terrorist attack application. All cities are attacked with positive probability. Amman, Beirut and Dubai, Istanbul are all directly connected to Baghdad. The city with the lowest value that has direct connection to New York is Istanbul, the city with lowest value that has a connection to London is Amman and the city with lowest value that has a connection to Paris is Beirut. Hence the attack path tree is given by the following graph.



Using the system of equations (4-7), the equilibrium attack probabilities are given by:

Amman	2.88%
Beirut	5.47%
Dubai	10.17%
Istanbul	7.89%
London	22.20%
New York	29.57%
Paris	21.81%

We note that the three cities of London, New York and Paris are all attacked with probability greater than 20%, whereas the four cities in the Middle east are attacked with probabilities at most equal to 10%. Interestingly, even though the difference in target values between two consecutive cities in the Middle East is as small as the difference between London and Paris, the difference in attack probabilities is much larger: it is on average 2.43% whereas the difference between Paris and London is only 0.39%. This reflects the fact that cities in the Middle East also serve as stops on the path to other targets, which magnifies the effect of differences in defense investments, and results in larger differences in attack probabilities.

The difficult step in the characterization of equilibrium is not the identification of the attack paths nor of the equilibrium distribution, but the characterization of the support of attacks Δ. However, the support can easily be derived when the underlying network is a line.

Proposition 2. *Let $\mathcal{G}$ be a line and $b_1, ..., b_n$ the increasing sequence of targets along the line. Let b_i be the first target such that*

$$\sum_{j=i+1}^{n} (1 - \frac{b_{j-1}}{b_j}) \frac{b_{j-1}}{b_i b_j} \leq 1$$

Then the equilibrium support is $\Delta = \{b_i, ..., b_n\}$.

Proposition 2 identifies the first target in a line. If $b_n(1-b_n) \leq b_{n-1}$, then there is no value $i < n$ such that $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$. By Proposition 1, the equilibrium is a pure strategy equilibrium. Otherwise, if $b_n(1-b_n) < b_{n-1}$, then there exists a unique target i which is the first target such that $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$, and hence the first target in the equilibrium attack path. Interestingly, the condition identifying target i as a first target does not only depend on the value of the target, but on the value of all subsequent targets in the path. The identification of the first target thus requires the recursive computation of the sum $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j}$ for all targets i in the increasing subsequence.

One case of interest is the case where the difference in values among two consecutive targets along the line is equal. Suppose $b_i = i\epsilon$ with $0 < \epsilon < \frac{1}{n}$. The value of the first target is then given by the first value i such that

$$(1+\epsilon)i + H_{i+1} \geq n - H_n,$$

where $H_j = \sum_{k=1}^{j}\frac{1}{k}$ is the jth harmonic number.

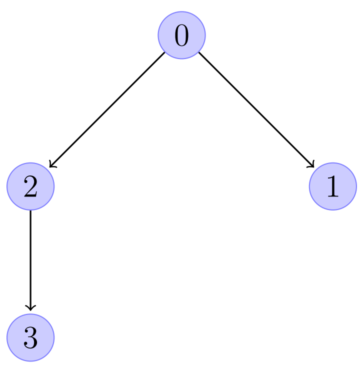
5 Comparative Statics

In this section, we discuss the consequences of changes in the parameters of the model on the equilibrium outcomes. We consider two comparative statics exercises. We first analyze the effect of an addition of a *single* link to the original network $\mathcal{G}$. We then analyze the effect of an increase in the value attached to a *single* node. In both cases we consider "small" changes, such that the support of the equilibrium attack distribution does not change.

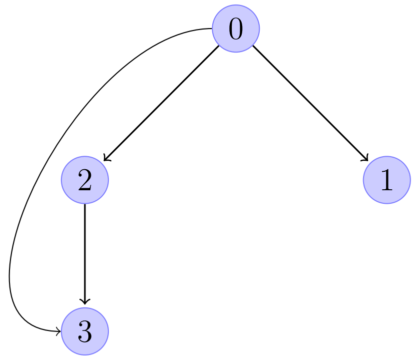
5.1 Adding a Link

We first look at the consequences of adding a link ij to the original network $\mathcal{G}$, restricting attention to the case where the link is added either between two targets i and j in the support which are previously unconnected, or between the attacker's node 0 and a target node i. Without loss of generality, if i and j are in the support, suppose that the value of target i is smaller than the value of target j, $b_i < b_j$. We first observe that the addition of a link may lead to a change in the attack support.

Example 1. Consider the following network with three targets:



Let the value of the targets be $b_1 = 0.275, b_2 = 0.3, b_3 = 0.4$. It is easy to check that in equilibrium, all three targets belong to the attack support and are attacked with probabilities $q_1 = 0.02, q_2 = 0.29, q = 0.69$. Now suppose that we add a link between 0 and 3 to the original graph.



After the addition of a direct link between the attacker and target 3, target 1 is no longer attacked, and the attacker's equilibrium attack distribution is given by $q_2 = 0.24, q_3 = 0.76$.

As shown in the proof of Theorem 2, for a fixed support, there is a unique equilibrium attack path tree. Either the addition of the new link does not affect the equilibrium path tree, in which case the equilibrium is identical to the equilibrium with the original network, or the addition of the new link affects the equilibrium attack path tree. In the latter case, it must be that the attack path between i and j uses the new link ij. By the construction of the equilibrium attack path tree, it must be that $b_i < b_k$ where k was the immediate predecessor of j in the attack path tree of the original network $\mathcal{G}$. Furthermore, all other nodes must remain connected to the same nodes as in the original network $\mathcal{G}$ and hence have the same immediate predecessors.

We first consider a situation where both i and j belong to the support and $b_i < b_k < b_j$ where k is j's immediate predecessor in the equilibrium attack path tree of the initial graph $\mathcal{G}$.

Proposition 3. *Consider the addition of a link between two points in the support, i and j with $b_i < b_j$. Then x_j goes up, and all x_l remains constant for any node l in $\Delta \setminus \Delta_0$. In addition*

- *If $b_i + b_k > b_j$, then U goes up, x_l goes down for any node l in Δ_0, and the attack probability q_j goes up while all other attack probabilities q_l go down.*
- *If $b_i + b_k < b_j$, then U goes down, x_l goes up for any node l in Δ_0, and the attack probability q_j goes down while all other attack probabilities q_l go up.*

Next, we consider the addition of a link between the attacker and a node i in the support where k is i's immediate predecessor of i in the original network $\mathcal{G}$.

Proposition 4. *Consider the addition of a link between a node in the support, node j, and the attacker. Then for all nodes $j \in \Delta \setminus \Delta_0$, x_j remains constant. In* *addition, if $b_j > 2b_k$, U goes down, x_l goes up for all first targets l in the original graph, and the probabilities q_l go up for all nodes $l \neq j$, while the probability q_j goes down.*

Propositions 3 and 4 show that the addition of a new link can lead to an increase or a decrease of the equilibrium utility of the attacker depending on the values of the targets. The fact that the addition of a new link may hurt the attacker is at first glance counterintuitive since the attacker could choose not to use the new link. However, the very existence of the new link changes the incentive structure of the problem, modifying the incentives for the attacker and the equilibrium defenses of the targets. This result is reminiscent of the Braess paradox in transportation economics, [2] where congestion may increase with the addition of a new link in the transportation network. As in the case of congestion in transportation networks, the equilibrium behavior of all players is affected by the change in the infrastructure network, causing the targets to increase their defense spending and making the attacks less likely to succeed.

Propositions 3 and 4 also indicate that a new link between i and j hurts the attacker when the value of the target, b_j, is large, and results in an increase in the attacker's utility when the value of the target is low. More precisely, the addition of a link between two nodes in the support increases the utility of the attacker when the sum of the values of the two predecessors of j before and after the addition of the link is greater than the value b_j ; it decreases the utility in the opposite situation, when the value b_j is higher than the sum of the values of its predecessors. When the new link connects the attacker to a node in the support, the situation is less clear. We provide a sufficient condition, showing that if b_j is greater than $2b_k$, the addition of a new link results in a decrease in the utility of the attacker.

The results of Propositions 3 and 4 can be illustrated in a line with increasing sequence $b_1 < ... < b_n$. For any target b_j, there exists a threshold value b_k such that the addition of a new link between b_j and $b_i > b_k$ results in an increase in the utility of the attacker whereas the addition of a new link between b_j and $b_i < b_k$ reduces the utility of the attacker. When a new link between b_j and 0 is formed, the utility of the attacker goes down whenever $b_j > 2b_k$.

Propositions 3 and 4 also analyze the effect of the addition of a new link on the attack probabilities. When the utility of the attacker goes up, the attacker increases his attack probability at node j and reduces it at all other nodes. By contrast, when the utility of the attacker goes down, the attacker decreases his attack probability at node j and increases it at all other nodes.

In order to study the effect of the addition of the new link on the equilibrium payoff of the targets, we need to sign the effect on the equilibrium defense investments, as payoffs are monotonically decreasing in the defense investments. We first note that, in all circumstances, the utility of target j goes down after the addition of a new link. Other nodes in $\Delta \setminus \Delta_0$ are not affected by the addition of the link. First targets in Δ_0 can either experience an increase or a decrease in utility, and the effect is negatively correlated with the effect on the attacker's utility. When the addition of the link increases the attacker's utility, it reduces the utility of the first targets ; when it decreases the attacker's utility, it leads to a higher utility for the first targets.

5.2 Change in value of a target

We next analyze the effect of an increase in the value b_i for some target $i \in \Delta$, assuming that the increase is small enough so as not to affect the support Δ nor the attack path tree $\mathcal{T}$. We distinguish between an increase in the value of a node which is not a first target and the increase in the value of a node which is a first target.

We first study the effect of an increase in the value of a node which is not a first target.

Proposition 5. *Consider an increase in b_i with $i \in \Delta \setminus \Delta_0$. Then x_i goes up, x_l goes down for all $l \succ i$ and x_j remains constant for all other nodes $j \notin \Delta_0$. In addition,*

- *If $\frac{b_{k(i)}(2b_{k(i)}-b_i)}{b_i^3} + \sum_{l,l \succ i} \frac{b_l - 2b_i}{b_l^2} > 0$, then U goes up, x_j goes down for all $j \in \Delta_0$, q_j goes down for all j which are not equal to i or immediate successors of i.*
- *If on the other hand $\frac{b_{k(i)}(2b_{k(i)}-b_i)}{b_i^3} + \sum_{l,l \succ i} \frac{b_l - 2b_i}{b_l^2} < 0$, then U goes down, x_j goes up for all $j \in \Delta_0$, q_j goes up for all j which are not equal to i or immediate successors of i.*

Next we analyze the effect of an increase in the value of a first target.

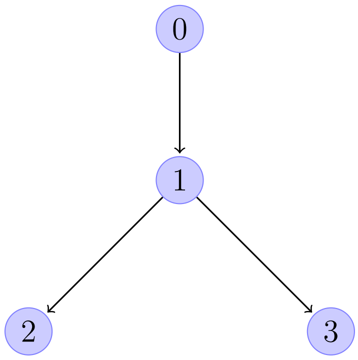
Proposition 6. *Consider an in increase in b_i with $i \in \Delta_0$. Then x_l goes down for all $l \succ i$ and x_j remains constant for all other nodes $j \notin \Delta_0$. In addition,*

- *If $\sum_{l,l \succ i} \frac{b_l - 2b_i}{b_l^2} > \frac{1}{8}$, then U goes up, x_j goes down for all $j \neq i \in \Delta_0$, q_j goes down for all j which are not equal to i or immediate successors of i.*
- *If $\sum_{l,l \succ i} \frac{b_l - 2b_i}{b_l^2} < -1$, then U goes down, x_j goes up for all $j \neq i \in \Delta_0$, q_j goes up for all j which are not equal to i or immediate successors of i.*

Propositions 5 and 6 show that, as in the case of the addition of a new link, an increase in the value of a target b_i can either increase or decrease the attacker's utility. The fact that the attacker is hurt when the value of the target increases may come as a surprise, but can easily be interpreted. A higher target leads to larger defense investments. Hence, the effect of an increase in the value of the target on the attacker's utility depends on the elasticity of the node's defense investments with respect to the target's value. If this elasticity is high, the attacker will be hurt by the increase in the target's value ; if the elasticity is low, it will benefit from that increase. In fact, even in the simplest case of a single node with value b_i, the equilibrium utility $U = b_i(1 - b_i)$ is non-monotonic in the value of the target, and an increase in b_i lowers the equilibrium payoff whenever $b_i > \frac{1}{2}$.

The same non-monotonicity of the attacker's equilibrium payoff with respect to the target's value arises in more complex networks, as shown in the following Example, which considers a star with three nodes.

Example 2. Consider a star network where 0 is connected to a hub 1 and 2 and 3 are peripheral nodes:



Let the hub values be given by $b_2 = 0.5, b_3 = 0.6$. The following graph shows how the expected utility U varies with b_1:

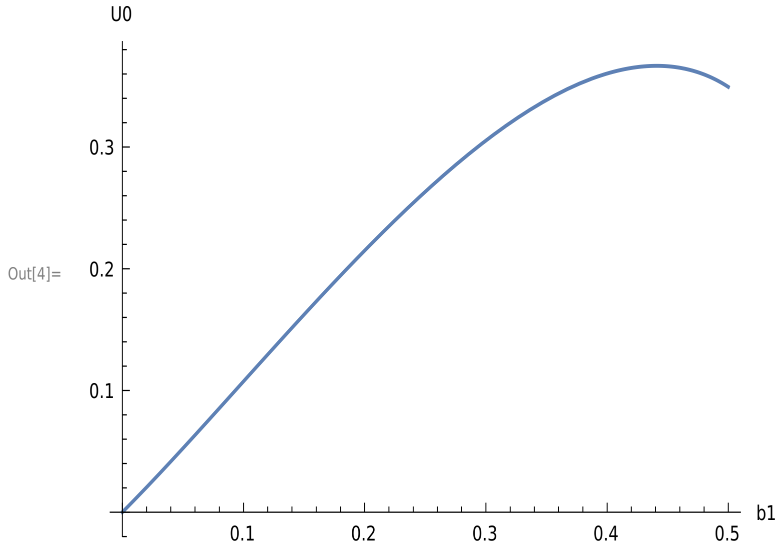


Figure 1: Attacker's utility as a function of b_1 in a star

The effect of an increase in b_1 on U_0 is not monotonic. For small values of b_1, an increase in b_1 increases the expected utility of the attacker, but that for a large value of b_1, an increase in b_1 reduces the expected utility of the attacker.

Propositions 5 and 6 generalize this intuition to general networks. When the value of an interior target (a target which is not a first target) increases, the effect on the attacker's utility depends on the value of the target, of its immediate predecessor and immediate successors. If the value of the target, compared to its immediate predecessor and successors, is relatively small, the attacker's equilibrium utility increases ; if the value of the target is relatively large, the attacker's equilibrium utility decreases. For a first target, the effect on the attacker's equilibrium utility depends on the difference between the value of the target and its immediate successors. Again, the attacker's equilibrium utility increases when the value of the target is low and decreases when the value of the target is high. As in the previous comparative statics exercise, when the change involves a first target, there exists a region of parameters for which we cannot ascertain the sign of the effect of the change in the target's value on the attacker's equilibrium utility.

We provide an illustration of the condition of Proposition 5 in a line with increasing sequence $b_1 < .. <, b_n$. Consider the effect of an increase in the value of target b_i with $i \geq 2$. Let $b = b_i$, $b_{i-1} = b - \delta$ and $b_{i+1} = b + \epsilon$. We compute

$$\begin{aligned}\frac{b_{i-1}}{b_i}\frac{2b_{i-1} - b_i}{b_i^2} + \frac{b_{i+1} - 2b_i}{b_{i+1}^2} &= \frac{(b-\delta)(b-2\delta)}{b^3} + \frac{\epsilon - b}{(b+\epsilon)^2}, \\ &= \frac{(b-\delta)(b-2\delta)(b+\epsilon)^2 - (b-\epsilon)b^3}{b^3(b+\epsilon)^2}.\end{aligned}$$

Hence the sign of $\frac{\partial q}{\partial b_i}$ is the same as the sign of

$$B = (b-\epsilon)b^3 - (b-\delta)(b-2\delta)(b+\epsilon)^2.$$

We check that B is increasing in δ. For $\delta = 0$, $B = -3\epsilon b - \epsilon b^2 < 0$. For $\delta = \epsilon$, $B = 3\epsilon^2 b + 2\epsilon^3 > 0$. Hence, there exists $\bar{\delta}(\epsilon, b) \in [0, \epsilon]$ such that

- $\frac{\partial U}{\partial b_i} > 0$ if $\delta \leq \bar{\delta}(\epsilon, b)$
- $\frac{\partial U}{\partial b_i} < 0$ if $\delta \leq \bar{\delta}(\epsilon, b)$

Note in particular that if the difference in values between b_{i+1} and b_i and b_i and b_{i-1} are equal, $\delta = \epsilon$, then $\frac{\partial U}{\partial b_i} < 0$, so an increase in b_i results in a decrease in the expected payoff for the attacker.

Propositions 5 and 6 also show that an increase in the value of the target can have different effects on the equilibrium attack probabilities. Whenever the attacker's expected utility increases, the attack probabilities of all nodes but node i and its immediate successors go down ; whenever the attacker's expected utility increases, the attack probabilities of all nodes but node i and it immediate successors go up. Note however that this result only signs the effect of an increase in b_i on the sum of attack probabilities on i and its immediate successors. If node i is not a final target, one cannot precisely ascertain the sign of the effect of an increase in b_i on the attack probability on node i.

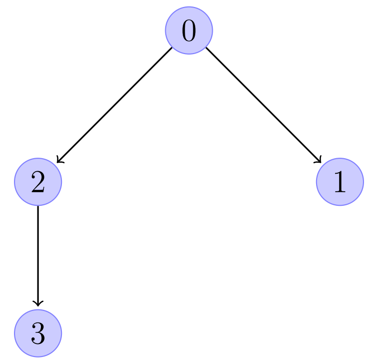
Turning to the expected payoff of the defense nodes, Propositions 5 and 6 show that all immediate successors of node i benefit from the increase in the value b_i whereas other interior nodes $j \neq i$ are unaffected by the change. First targets $j \neq i$ can either experience a decrease in equilibrium payoff (when the attacker's expected utility increases) or a decrease in equilibrium payoff (when the attacker's expected utility decreases). The effect of an increase in b_i on the equilibrium utility of node i is always negative when i is an interior target, but can either be positive or negative when i is a first target.

6 Sequential game

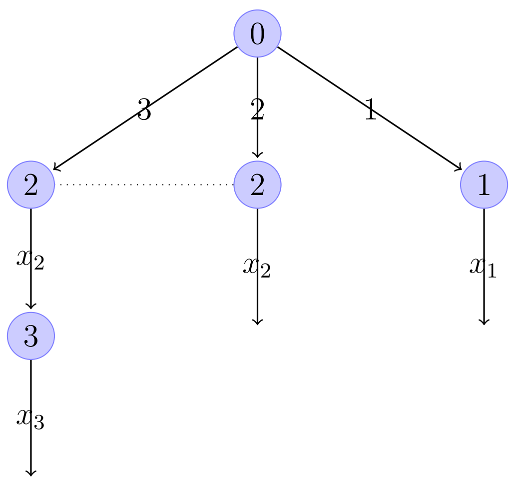
In the benchmark model, we implicitly assume that there are fixed costs of setting up an interception protocol, and all targets decide simultaneously at the beginning of the game whether to incur these costs. Hence nodes commit to an interception strategy, and Nash equilibrium is the appropriate solution concept.

We now consider a different version of the game, where targets incur variable costs of interception, and decide how much to spend on defense upon observing the arrival of a suspicious package. In order to make the analysis more transparent, we assume that there is no fixed cost. The game is now a sequential game where (i) first, the attacker chooses a path and a target and (ii) every node chooses how much to invest in interception when it observes a suspicious package. We assume again a quadratic cost of interception, $c(x_i) = \frac{1}{2}x_i^2$.

To illustrate the difference between the simultaneous and sequential games, consider the following network with three possible targets:



In the benchmark game, the attacker chooses a target in $\{1, 2, 3\}$ and each defender simultaneously chooses an interception level x_1, x_2, x_3. The sequential game is the extensive-form game pictured as follows:



Notice that target 2 chooses an interception level on two branches of the extensive-form game: when it is attacked and when it lies on the attack path to target 3. The defender at node 2 is however unable to distinguish between the two situations, as the only observation it makes is the arrival of the suspicious package. As the attacker will only choose one attack path, targets in general may be left off the equilibrium play.

We use as our equilibrium concept the sequential equilibrium of Kreps and Wilson [15]. For completeness, we give below the definition of sequential equilibrium. Consider a finite game in extensive form G with players $i = 0, 1, 2..n,$, $h_i^j \in H_i$ is the j^{th} information set for Player $i \in \{1, 2, ...n\}$, $A(h_i^j)$ the actions available to Player i at h_i^j and x_i^j a decision node for i in h_i^j. Let β_i be a behavioral strategy for player i with $\beta_i^j \in \Delta A(h_i^j)$ a probability distribution over the actions available to i at h_i^j.

Let $\beta_{i\varepsilon}$ be a behavioral strategy that places a positive probability of at least $\varepsilon^k > 0$ on every action in every information set. Let $\mu_\varepsilon(x_i^j \mid h_i^j)$ be the probability of being at node x_i^j, derived by Bayes' Theorem from the strategies β, if information set h_i^j is reached.

An assessment (μ, β) is a *sequential equilibrium* if the following conditions hold:

1. **Sequential rationality.** The behavioral strategy β_i is a best response to β_{-i} at every information set, given beliefs μ.

2. **Consistency.** There exist $(\beta_{i\varepsilon}, \mu_{i\varepsilon})$ such that for all i, $\lim_{\varepsilon \to 0}((\beta_{i\varepsilon}, \mu_{i\varepsilon}) = (\beta_i, \mu_i)$

The use of this solution concept is necessary for two reasons. A target i might be on the equilibrium path of the attacker but if the package arrives at i, this target knows its predecessors have not been attacked. It must therefore update, by Bayes' theorem, its probability of being a target. The second factor arises if player i is not on the equilibrium attack path but the package arrives there. What can be inferred about the probability of being attacked? To answer this question, we perturb the actions at every information set to obtain a completely mixed strategy and use Bayes' theorem to calculate beliefs. We then take the limit of these beliefs as the perturbations go to 0 and use these beliefs for evaluating the players' behavioral strategies. Note that there could be many different perturbations, and we only need to select one to justify the beliefs.

It is well-known that sequential equilibria always exist (see Kreps and Wilson [15] Proposition 1 p. 876). We first prove that in any sequential equilibrium the attacker must attack a single target under an additional assumption on target values. Like our previous Assumption (Assumption 1) stating that all target values are different, the following Assumption holds generically for all target values $(b_1, ..., b_n)$.

Assumption 2. *For all* i, j ,

$$b_i(1 - b_i) \neq b_j$$

and

$$b_i(1 - b_i) \neq b_j(1 - b_j).^3$$

[3]Note that the second part of Assumption 2 implies Assumption 1.

Theorem 3. *Suppose that Assumption 2 holds and that $\mathcal{G}$ is a tree. Then in any sequential equilibrium, the attacker attacks a single target with probability one.*

The intuition underlying this result is easy to grasp. The last target i along an attack path updates her belief and thinks that she will be attacked with certainty, thereby spending $x_i = b_i$ in interception. Hence, the expected payoff of the attacker at any final target is equal to $\alpha_i b_i(1 - b_i)$. If the attacker attacks another target j immediately before i on the equilibrium path to i, it will obtain an expected payoff of $\alpha_j b_j(1 - b_j)$. However, as $\alpha_i = \alpha_j(1 - b_j)$, for the attacker to be indifferent between the two targets, the equality $b_j = b_i(1 - b_i)$ must hold, contradicting Assumption 2. By a similar reasoning, if the attacker attacks two targets i and j along different paths, the equality $b_i(1-b_i) = b_j(1-b_j)$ must hold, violating again Assumption 2.

Theorem 3 shows that, by contrast to the simultaneous game, all sequential equilibria are equivalent to an equilibrium in pure strategies: the attacker attacks a single target, which is the only node choosing positive interception expenses along the equilibrium path. Hence, allowing for sequential observation of the arrival of suspicious objects and letting defenders update accordingly their beliefs over the attacker's strategy makes it impossible for the attacker to randomize over targets in equilibrium.

Our next result characterizes the sequential equilibria when the underlying network $\mathcal{G}$ is a tree . We first establish the following Lemma. Define a *branch* to be a sequence of nodes from node 0 to any leaf of the tree. Let $\mathcal{B}$ denote the set of branches and $B \in \mathcal{B}$ denote a branch. Of course, there can be $i \in B \cap B'$.

Let $P(i, B), S(i, B)$ denote respectively the predecessors and successors of i in B. [4]

Lemma 5. *Consider any branch of $\mathcal{G}$, labeled $\{, 1, 2 \ldots, K\}$ for convenience. There exists $i \in \{1, ...K\}$ such that*

$$
\begin{aligned}
b_i(1 - b_i) &\geq b_j \ \forall j < i && (9)\\
b_i &\geq \Pi_{k=1}^{j-i}(1 - b_{i+k})b_j \ \forall j \in \{i + 1, \ldots, K\} && (10)
\end{aligned}
$$

Theorem 4. *Suppose $\mathcal{G}$ is a tree. Then, i belonging to any branch $B = \{1, \ldots, K\}$ is a target attacked in a pure strategy sequential equilibrium if and only if it satisfies*

(i) Equations (9) and (10) for branch B.

(ii) for all $j \in B' \in \mathcal{B}$, $B' \neq B$,

$$b_i(1 - b_i) \geq \Pi_{k \in P(j,B')}(1 - b_k)(1 - b_j)b_j \tag{11}$$

The proof specifies out-of-equilibrium beliefs that *all* nodes not on the equilibrium path will, in the event of the suspicious parcel reaching its location, believe with probability one that they are under attack. This implies that any deviation

[4] The set of predecessors of i can be defined without reference to the branch B. But, if $i \in B \cap B'$, then the set of successors in B will be different from those in B'.

will be checked to the maximum extent possible, and hence result in the lowest possible expected payoff following a deviation. This results in a weakest possible sufficient condition. The conditions specified in the theorem also turn out to be necessary.

A simple construction shows how to locate a node satisfying the three conditions. Let

$$I = \{i | i \text{ satisfies equations 9 and 10 for some } B \in \mathcal{B}\}$$

.

Let $i^* = \text{argmax}_{i \in I}(1 - b_i)b_i$. Lemma 5 shows that I is nonempty and Assumption 2 establishes that i^* is well-defined. It is now elementary to show that i^* satisfies equation (11). For suppose j lies on a different branch B' . Let $k \in B' \cap I$. Then, since $b_{i^*}(1 - b_{i^*}) > b_k(1 - b_k)$ and k satisfies equations (9 and (10) with respect to j, i^* must satisfy equation (11) with respect to j.

Theorem 4 characterizes the nodes which can be attacked in a sequential equilibrium. The following example shows that for the same vector of target values $(b_1, ..., b_n)$ on a line, different nodes can be attacked in a sequential equilibrium.

Example 3. Suppose there are three locations with b_i's being given by the vector $(\frac{1}{5}, \frac{2}{9}, \frac{1}{2})$. It is easy to check that both b_1 and b_3 satisfy equations (9) and (10). Hence there exists an equilibrium where the attacker attacks target 1 because $b_1 > (1-b_2)b_2$ and $b_1 > (1-b_2)(1-b_3)b_3$. However, there exists also an equilibrium where the attacker attacks target 3 because $b_1 < b_3(1 - b_3)$ and $b_2 < b_3(1 - b_3)$.

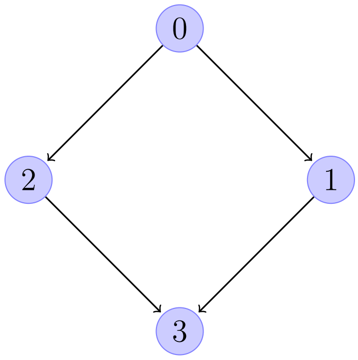
Example 3 thus shows another striking difference between the simultaneous and the sequential games. While the Nash equilibrium of the simultaneous game is always unique, the sequential game may admit multiple sequential equilibria, supported by different beliefs. In example 3, target 1 is attacked in an equilibrium supported by both targets 2 and 3 believing that they are attacked with probability close to 1 off the equilibrium path. Target 3 is attacked in an equilibrium where both preceding nodes believe, on the equilibrium path, that they will not be attacked.

7 Cooperation in defense

In the benchmark model, each of the nodes chooses independently the level of defense. We now consider an extension where all nodes cooperate in defense. The game then becomes a two-person game played between an attacker A and a single defender D whose objective is to minimize the sum of losses of all the nodes.

A major difference between the cooperative situation and the benchmark, decentralized, model is that the single defender D protects all nodes on attack paths, including nodes outside the support, as she internalizes the positive externality of defense of a node on all the targets attacked through that node. Furthermore, because nodes outside the support are protected, the attacker no longer selects a unique path to attack a given target. As the following example shows, in equilibrium, targets can be attacked through different paths.

Example 4. Consider a network with three nodes, where node 3 can be attacked along two different paths, as illustrated below:



Assume that $b_1 = 0.5$ and $b_3 = 0.8$. For different values of b_2, the following are equilibria of the game of attack and interception with cooperation in defense:

- If $b_2 \leq \frac{4}{17}$, D chooses defense investments $y_1 = y_2 = 0.11$ and $y_3 = 0.70$, and A attacks node 3 with probability 1.

- If $\frac{4}{17} < b_2 \leq \frac{4}{7}$, D chooses defense investments satisfying $y_1 = y_2 = \frac{b_2}{2}, y_3 = \frac{4-5b_2}{4}$. A attacks target 2 with probability $r_2 = \frac{17b_2-4}{8(2-b_2)}$, attacks 3 through node 1 with probability $r_{31} = \frac{1}{2}$ and attacks 3 through node 2 with probability $r_{32} = \frac{3(7b_2-4)}{8(2-b_2)}$

- If $\frac{4}{7} < b_2 \leq \frac{4}{5}$, D chooses defense investments $y_1 = y_3, y_2$. A attacks target 2 with probability r_2 and target 3 through node 1 with probability $r_3 = 1 - r_2$.

This example shows that as long as $b_2 < \frac{4}{7}$, the attacker uses the two paths $p = 1,3$ and $p' = 2,3$ to attack target 3. Furthermore, for intermediate values $\frac{4}{17} < b_2 < \frac{4}{7}$, the attacker uses two different paths where the set of predecessors in the support are different. Along path $1,3$, there is no predecessor on the attack path, and along path $2,3$, node 2 is a target preceding 3 on the attack path.

Example 4 shows that Lemma 2 does not hold when nodes cooperate in defense. The attack paths do not necessarily form a tree, and the strategy of the attacker must prescribe a probability distribution not only over targets but also over paths leading to a target. The analysis of the game thus becomes more complex than when nodes independently choose their level of defense, and in order to characterize equilibrium, we restrict attention to lines.

Let us consider a line with an increasing sequence of target values $b_1 < ... < b_n$. Let $(y_1, ..., y_n)$ be the defense investments, Δ the support of equilibrium, and r_i the probability of attack of a target $i \in \Delta$. The objective of the defender is to minimize the loss function

$$\mathcal{L} = \sum_{i \in \Delta} \prod_{k \leq i} (1 - y_k) r_i b_i + \sum_{i \in N} \frac{y_i^2}{2}.$$

The best response of the defender is given by the first order conditions:

$$y_i = \prod_{k<i}(1-y_k)\sum_{j\in\Delta, j\geq i}\prod_{i<l\leq j}(1-y_l)b_j r_j,$$

for all $i \in N$.

We first show that equilibrium defense investments are decreasing along the line.

Lemma 6. *For all $i = 1, \ldots, y_n$, $y_i \geq y_{i+1}$ with equality holding iff $r_i = 0$.*

The intuition underlying Lemma 6 is easy to grasp. Because all attack paths follow the line, for any probability distribution of the attacker, the expected loss at node i is larger or equal to the loss at node $i+1$, strictly larger if node i is attacked with positive probability. Hence the defender puts a larger defense on node i than node $i+1$. The next Proposition shows that, as in the benchmark model, the equilibrium support and probabilities are unique when nodes cooperate in defense on the line.

Proposition 7. *Suppose that $\mathcal{G}$ is a line. Then, there is a unique equilibrium of the game where nodes cooperate in defense.*

Our next results compare the equilibria of the game where nodes cooperate in defense and when defense investments are chosen independently. We first compare the equilibrium supports. Proposition 2 characterizes the equilibrium support in the decentralized model. In Appendix B, we provide an algorithm to similarly construct the support in the cooperative model. According to Lemma 3, in the benchmark model, all nodes in the increasing sequence following the first target are attacked with positive probability. Example 5 shows that this is not necessarily the case when the nodes cooperate in defense.

Example 5. Consider a line with three nodes, labeled 1, 2, 3 with $b_1 = 0.5$, $b_3 = 0.8$, $0.5 < b_2 < 0.8$. In the noncooperative case, all three nodes belong to the support. In the cooperative case, using the algorithm of Appendix B, $\xi_1 = 0.25$ so that $i^* = 1$; the first node attacked is node 1. Next we compute $\zeta_2(1) = 1 - \frac{1}{2b_2}$ and $\zeta_3(1) = 1 - \sqrt{\frac{5}{8}}$. If $b_2 < \sqrt{\frac{2}{5}}$, then $\zeta_2(1) < \zeta_3(1)$ and node 2 is not attacked. If $b_2 > \sqrt{\frac{2}{5}}$, then $\zeta_2(1) > \zeta_3(1)$ and all nodes are attacked.

Example 5 shows that in the cooperative model, there can be "gaps" in the support of equilibrium: the attacker attacks nodes 1 and 3 but skips node 2 even though the values $b_1 < b_2 < b_3$ for an increasing sequence . In Example 5, the equilibrium support is larger in the noncooperative model than in the cooperative case. This may not always be the case, as shown in the next example:

Example 6. Consider a line with 6 nodes with $b_1 = 0.2, b_2 = 0.21, b_3 = 0.22, b_4 = 0.23, b_5 = 0.24$ and $b_6 = 0.5$. In the benchmark model, as $b_6(1-b_6) = 0.25 > b_i$ for all $i \leq 5$, by Proposition 1, the only equilibrium is a pure strategy equilibrium where the attacker attacks node 6 with probability 1. When nodes cooperate in defense, using the algorithm of Appendix B, we compute $\xi_1 = 0.2, \xi_2 = 0.174, \xi_3 = 0.156, \xi_4 = 0.144, \xi_5 = 0.135, \xi_6 = 0.183$, so $i^* = 1$: the first node attacked in the support is node 1. Next we compute $\zeta_2(1) = 0.048, \zeta_3(1) = 0.302, \zeta_4(1) = 0.507, \zeta_5(1) = 0.634, \zeta_6(1) = 0.902$, so $j^*(i^*) = 6$ and node 6 is the second (and last) node attacked in equilibrium.

Examples 5 and 6 show that there is no general inclusion result comparing the equilibrium support in the cooperative and noncooperative models. In our next Proposition, we provide some welfare comparisons between the cooperative and noncooperative models. The comparison assumes that *all* nodes are in the support of the attacker's equilibrium strategy in both the cooperative and noncooperative cases.

In order to avoid confusion, let $\beta_i^c = (1-y_1)\dots(1-y_i)$, and $\beta_i = (1-x_1)\dots(1-x_i)$,

Also, let $\mathcal{L}_i = \beta^c b_i r_i + \frac{y_i^2}{2}$ denote the equilibrium expected loss suffered by node i in the cooperative case, and $\mathcal{M}_i = \beta_i b_i q_i + \frac{x_i^2}{2}$ be the equilibrium expected loss suffered by node i in the cooperative case, and $\mathcal{M} = \sum_{i=1}^n \mathcal{M}_i$.

Proposition 8. *Let $\mathcal{G}$ be a line on n nodes and suppose that all n nodes are attacked in both the cooperative and non-cooperative cases. Then, the following are true.*

(i) $\mathcal{M}_i > \mathcal{L}_i$ *for* $i = 2, 3 \dots, n-1$ *and* $\mathcal{M}_n = \mathcal{L}_n$.

(ii) $\mathcal{M} > \mathcal{L}$

(iii) A's expected payoff is strictly higher in the benchmark model.

This result shows that nodes 2 to $n-1$ always benefit from coordinated defense, while node n 's expected loss is the same under both the centralized and decentralized defense cases. The attacker always prefers the decentralized defense scenario whereas the defenders collectively prefers the centralized scenario.

The only player whose welfare cannot easily be compared in the two cases is the first target, node 1. The node is more heavily defended in the cooperative case and so is less heavily attacked. But node 1 also spends more on defense and this increases its cost of defense. The next example shows that the comparison can go in both directions, depending on the values of the targets.

Example 7. As in example 5, consider again three nodes on a line with $b_1 = 0.5, b_3 = 0.8$. First, assume that $b_2 = 0.65$.

Using $U = (1 - q_1 b_1)b_1$ and equation 7

$$1 - q_1 = q_2 + q_3 = \frac{(b_2 - b_1)}{b_2^2} \frac{1}{(1 - q_1 b_1)} + \frac{(b_3 - b_2)}{b_3^2} \frac{b_2}{(1 - q_1 b_1)b_1}$$

This quadratic equation in q_1 can be solved to yield $q_1 = 0.247$. Hence,

$$\mathcal{M}_1 = 0.116$$

On the other hand, $r_1 = (\frac{1-b_2}{b_2})^2 = 0.29$ and

$$\mathcal{L}_1 = 0.197$$

So, $\mathcal{M}_1 < \mathcal{L}_1$. However, choose now $b_2 = 0.78$. Then, $r_1 = 0.0795$, $q_1 = 0.374$ and

$$\mathcal{M}_1 = 0.170 > 0.145 = \mathcal{L}_1$$

So, node 1's expected loss in the cooperative case can be either higher or lower than in the non-cooperative case.

Example 7 thus shows that the first target can be asked to increase its interception level in the cooperative case to a point which decreases her payoff compared to the decentralized scenario. This opens up the possibility that other nodes have to subsidize node 1 (in case $\mathcal{L}_1 > \mathcal{M}_1$) in order to establish coordinated defense. Of course, part (ii) of Proposition 8 shows that the aggregate benefits of nodes 2 to $n-1$ are sufficiently large to more than compensate node 1's loss if any.

8 Conclusions

This paper studies a game of attack and interception in a network, where a single attacker chooses a target and a path, and each node chooses a level of protection. We show that the Nash equilibrium of the game exists and is unique. It involves a mixed strategy of the attacker except when one target has a very high value relative to others. We characterize equilibrium attack paths and attack distributions as a function of the underlying network and target values. We also show that adding a link or increasing the value of a target may harm the attacker - a comparative statics effect which is reminiscent of Braess's paradox in transportation economics. Finally, we contrast the Nash equilibrium with the equilibria of two variations of the model: one where nodes make sequential protection decisions upon observing the arrival of a suspicious object, and one where all nodes cooperate in defense.

While we treat in the analysis the network as exogenous, there are clearly situations where the defenders can design the network to prevent or fend off attacks. In computer networks, the defenders can choose the architecture of the network. In infrastructure networks, the defenders can block or eliminate edges, as considered for example in the early literature on network interdiction. The analysis of endogenous network design by defenders whose objective is to promote communication while preventing attacks by adversaries is an important topic for future research.

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Appendix A: Proofs

Proof of Lemma 1

Proof. Both equalities follow from the fact that the attacker must be indifferent between attacking any two nodes in Δ. The first equality stems from the observation that if the attacker attacks the two targets i and j, he must receive the same expected payoff by attacking i and j

$$b_i \beta_i(p) = b_j \beta_j(p),$$

but because j and i are successive targets along path p,

$$\beta_i(p) = \beta_j(p)(1 - x_i),$$

yielding the result.

The second equality stems from the fact that if i and j are the first targets on two paths p and p', then $\alpha_i(p) = \alpha_j(p') = 1$, so that indifference implies

$$b_i \beta_i(p) = b_i(1 - x_i) = b_j \beta_j(p') = b_j(1 - x_j).$$

. □

Proof of Lemma 2

Proof. Let i be the first target which is attacked along the paths p and p' and has two different predecessors, $j \in p \cap \Delta$ and $k \in p' \cap \Delta$. From the previous Lemma, as j and k are both immediate predecessors of i, we must have

$$b_j = b_i(1 - x_i) = b_k,$$

contradicting the fact that $b_j \neq b_k$ for two different defenders j and k. □

Proof of Theorem 1

Proof. Reinterpret the game as a game with continuous strategy spaces, where the attacker chooses a point q in the $n-1$ dimensional simplex and every defender i chooses an investment $x_i \in \Re_+$. We will use the Debreu-Fan-Glicksberg Theorem (Debreu [6], Fan [9] and Glicksberg [12]) to prove existence of an equilibrium in pure strategies of this game.

First note that because $b_i \leq 1$, the strategy space of defender i can be restricted to $[0, 1]$, so that the strategy spaces of all players are compact, convex subsets of Euclidean spaces. Second, an immediate inspection of equations (1) and (2) shows that the payoffs of the players are continuous in the product of the strategies $(q, x_1, ..., x_n)$. The payoff of the attacker given by (1) is linear and hence quasi-concave in q. Given that the quadratic cost function is convex, the payoff of any defender i given by (2) is a concave function of x_i and hence is quasi-concave. All assumptions of the Debreu-Fan-Glicksberg Theorem are thus satisfied, and the game admits an equilibrium in pure strategies, which is a mixed strategy Nash equilibrium of the original game of attack and interception. □

Proof of Proposition 1

Proof. Suppose that there exists a defender i whose value satisfies the condition. Consider the strategy profile where the attacker chooses $q_i = 1$, and some path to i, defender i chooses $x_i = b_i$ and all defenders $j \neq i$ choose $b_j = 0$. The expected payoff to the attacker is $U_0 = b_i(1 - b_i)$.

Suppose 0 deviates and chooses q' such that $q'_j > 0$ where $j \neq i$ as well as some attack path p to j. We know that $x_j = 0$. If this is to be a profitable deviation, then we might as well set $q'_j = 1$.

If $i \notin P(p, j)$, then

$$U_0(q', x) = b_j \leq b_i(1 - b_i)$$

If $i \in P(p, j)$, then

$$U_0(q', x) = b_j(1 - b_i) < b_i(1 - b_i) \text{ since } b_i > b_j$$

Any defender $j \neq i$ is attacked with probability 0 and hence optimally chooses not to invest in the detection technology. Finally defender i chooses x_i to maximize

$$U_i = -b_i(1 - x_i) - \frac{1}{2}x_i^2,$$

resulting in the optimal decision $x_i^* = b_i$.

Suppose now that the game admits an equilibrium in pure strategies where the attacker chooses $q_i = 1$. As we just argued, defender i then optimally chooses a detection probability $x_i^* = b_i$, resulting in an expected payoff $U_0 = b_i(1 - b_i)$ for the attacker. Moreover, for all $j \neq i$, $x_j^* = 0$. For any j such that there exists a path p such that $i \notin P(p, j)$, a deviation to j gives 0 a payoff of b_j. If for all paths p, $i \in P(p, j)$ then 0 gets a payoff of $b_j(1 - b_i)$ by deviating. This establishes the necessity of the conditions. □

Proof of Lemma 3

Proof. To prove the first statement, suppose by contradiction that $b_j \geq b_i$. Because j precedes i , $\beta_j = \alpha_j(1 - x_j^*) \geq \alpha_i > \alpha_i(1 - x_i^*) = \beta_i$. Hence

$$\beta_j b_j > \beta_i b_j \geq \beta_i b_i,$$

contradicting the fact that $i \in \Delta$.

To prove the second statement, suppose by contradiction that $j \notin \Delta$. Then, $x_j^* = 0$. Let $p = \{i_1, \ldots, i_K\}$ where (i) $i_1 = i, i_K = j$ and $i_k i_{k+1} \in \mathcal{G}$. Suppose for all $k \in p, k \neq i$, $x_k^* = 0$ since $k \notin \Delta$. Then, $\beta_i = \beta_j$.

However, because $j \notin \Delta$ and $i \in \Delta$, we must have

$$b_i \beta_i \geq b_j \beta_j,$$

resulting in a contradiction. □

Proof of Lemma 4

Proof. Suppose by contradiction that there is $j \in J$ with $b_j < b_k$. Since there is a path from j to i which does not intersect Δ, $x_l^* = 0$ for all targets l on this path that are distinct from j and i. Hence, by attacking i through j would ensure that

$$\beta_j = \alpha_i \tag{12}$$

Since k and j are both in Δ,

$$\beta_k b_k = \beta_j b_j.$$

So, $b_j < b_k$ implies that $\beta_j > \beta_k = \alpha_i$. But, equation (12) shows that the attacker can increase his payoff by attacking i through the path from j to i in $\mathcal{G}$. □

.

Proof of Theorem 2

Proof. : We first consider the case analyzed in Proposition 1 which describes the conditions under which there can be a pure strategy equilibrium with only i being attacked in equilibrium.

So, suppose there is i such that

(i) $b_i(1 - b_i) \geq b_j$ for all j such that $i \notin P(j)$.

(ii) $b_i \geq b_j$ for all j such that $i \in P(j)$.

Suppose by contradiction that there is an equilibrium (q, x^*) such that $q_j > 0$ for some $j \neq i$. Then, $x_i^* < b_i$.

Let $i \in P(j)$. Then, $\beta_i \leq \beta_j$ and $b_i > b_j$. So,

$$\beta_i b_i > \beta_j b_j$$

This shows that a deviation to attacking i with probability one must be profitable to i.

Next, suppose $i \notin P(j)$. Then, $\beta_i > (1 - b_i)$ and so a deviation to attacking i with probability one gives

$$\beta_i b_i > (1 - b_i) b_i \geq b_j > \beta_j b_j$$

This establishes uniqueness of equilibrium when the conditions for a pure strategy equilibrium are satisfied.

Next consider the case where there is no pure strategy equilibrium. The strategy of the proof is the following. We suppose that there are two equilibria E and E' with supports Δ and Δ'. We first prove that $\Delta = \Delta'$. We then show that the sequence of targets on equilibrium paths have to be equal in the two equilibria and finally establish that the equilibrium probabilities over the targets have to be equal.

Step 1: The support of targets are equal $\Delta = \Delta'$.

Suppose by contradiction that there exists $i \in \Delta$ such that $i \notin \Delta'$.

Claim 1. *The equilibrium utility in the two equilibria must satisfy: $U' > U$.*

Proof. Suppose first that there exists a path to i in $\mathcal{G}$ which does not intersect Δ'. By attacking i along the path, the attacker obtains a payoff b_i. As $i \notin \Delta'$, we must have $U' \geq b_i$. In addition, as i is attacked with positive probability in equilibrium E, $x_i^* > 0$ and $U = \alpha_i b_i (1 - x_i^*) < b_i$, establishing that $U' > U$.

Suppose next that all paths to i in $\mathcal{G}$ intersect Δ'. This is in particular true for the equilibrium attack path p to i in E. Let j be the last point in Δ' along path p. By Lemma 3, $b_j < b_i$. But then, as $i \notin \Delta'$, and there is a path between j and i which does not intersect Δ', by the second part of Lemma 3, $i \in \Delta'$, a contradiction. □

Claim 2. *The supports must satisfy:* $\Delta' \subset \Delta$.

Proof. Suppose by contradiction that there exists $i \in \Delta', i \notin \Delta$. Applying the same argument as in the proof of Claim 1, $U > U'$, contradicting the fact that $U' > U$. □

Claim 3. *For any* $i \in \Delta'_0$, $q'_i < \alpha_i q_i$.

Proof. Pick a target i which is a first target along some attack path in the equilibrium E'. As $\Delta' \subset \Delta$, $i \in \Delta$. Suppose first that i is a first target in equilibrium E as well, $i \in \Delta_0$. Then, by Claim 1,

$$U' = b_i(1 - x_i^{*'}) > U = b_i(1 - x_i^*),$$

and by equation (3) and the fact that $\alpha_i = \alpha'_i = 1$,

$$x_i^{*'} = q'_i b_i, x_i^* = q_i b_i,$$

yielding the result.

Next suppose that $i \in \Delta_m$ for some $m \geq 1$. Let p be the path in $\mathcal{T}$ from 0 to i in E. Consider the equilibrium path p' in $\mathcal{T}'$ from 0 to i in E'.[5]

Consider equilibrium E. If there is no node in Δ along the path p', then by deviating and attacking i along that path, the attacker obtains a payoff $b_i(1 - x_i^*) > b_i \alpha_i (1 - x_i^*)$ as $\alpha_i < 1$ because there is another node attacked with positive probability before i on path p. This shows that the path p' must intersect Δ.

Let j be the last point in Δ along the path p'. As $i \in \Delta'_0$, $j \notin \Delta'$. We consider two cases. Suppose first that j is on the equilibrium attack path p. Then there cannot be any other node between j and i in p. If there was a node k between j and i in p, the attacker would have a profitable deviation by attacking i directly from j along path p', resulting in an expected utility $b_i \beta_j (1 - x_i^*) > b_i \alpha_i (1 - x_i^*)$, as $\alpha_i < \beta_j$. Hence, j is the immediate predecessor of i along path p. By Lemma 1,

$$b_j = b_i(1 - x_i^*).$$

and by equation 3

$$x_i^* = \alpha_i q_i b_i.$$

[5] Of course, both paths p and p' are in $\mathcal{G}$.

Note that $j \notin \Delta'$ and there exists a path to j which does not intersect Δ'. Hence,

$$b_j \leq U' = b_i(1 - x_i^*),$$

with

$$x_i^{*'} = q_i' b_i.$$

showing that

$$q_i' < \alpha_i q_i.$$

Next suppose that j is not on the equilibrium attack path p to i and let k be the last target preceding i on the equilibrium path p. Because p' is a path from j to i which does not intersect Δ, by Lemma 4,

$$b_k < b_j.$$

By Lemma 1

$$b_k = b_i(1 - x_i^*),$$

so that

$$b_i(1 - \alpha_i q_i b_i) = b_k < b_j \leq b_i(1 - q_i' b_i),$$

This establishes

$$q_i' < \alpha_i q_i.$$

□

Claim 4. *Suppose that $m \geq 1$. Then for any $i \in \Delta_m'$, if j is the predecessor of i on the equilibrium path p' in E', j is also the predecessor of i on the equilibrium path p in E.*

Proof. Pick a target $i \in \Delta_m'$. We first claim that $i \notin \Delta_0$. Suppose to the contrary that $i \in \Delta_0$. Then there is a path to i which does not intersect Δ. As $\Delta' \subset \Delta$, the path does not intersect Δ' either. But then because $i \in \Delta_m'$ and $m \geq 1$, $b_i(1 - x_i^{*'}) > b_i \alpha_i' (1 - x_i^{*'})$, so that the attacker has a profitable deviation, establishing a contradiction.

So let $p \geq 1$ be the distance to i in equilibrium E. We claim that the immediate predecessor of i on the two equilibrium attack paths must be identical. Suppose by contradiction that j is the immediate predecessor of i on the equilibrium path p' and $k \neq j$ the immediate predecessor of i on the equilibrium path p

For all l on p' between j and i, as $l \notin \Delta'$ by the second part of Lemma 3 $b_l < b_j$. Hence by the first part of Lemma 3 as $b_j \in \Delta$, $b_l \notin \Delta$. Hence the subpath of p'

joining j and i does not intersect Δ. Because $\Delta' \subset \Delta$, the subpath of p joining k and i does not intersect Δ' either. But then, by Lemma 4 we obtain

$$b_j < b_k \text{ and } b_k < b_j,$$

a contradiction. Hence the predecessor of i on the two equilibrium attack paths p and p' must be identical. □

Claim 5. *For all $i \in \Delta'$, $q'_i < q_i$.*

Proof. By Claim 3, the statement is true whenever $i \in \Delta'_0$. Now consider $i \in \Delta'_m$ with $m \geq 1$. By Claim 4, i has a common set of predecessors $i_0, ..., i_{m-1}$ in the two equilibria.

By Lemma 1 for all $m \geq 1$,

$$b_{i_{m-1}} = b_{i_m}(1 - \alpha'_{i_m} q'_{i_m} b_{i_m}) = b_{i_m}(1 - \alpha_{i_m} q_{i_m} b_{i_m}),$$

so that

$$\alpha'_{i_m} q'_{i_m} = \alpha_{i_m} q_{i_m}.$$

Now recall that $\alpha'_{i_0} = \alpha_{i_0} = 1$. We now prove by induction that $\alpha'_{i_m} > \alpha_{i_m}$ for all $m \geq 1$. Let $m = 1$. By Lemma 3, $q'_{i_0} < q_{i_0}$ so that

$$\alpha'_{i_1} = 1 - b_{i_0} q'_{i_0} > 1 - b_{i_0} q_{i_0} = \alpha_{i_1}.$$

.

Consider then the inductive step. Suppose that $\alpha_{i_{m-1}} < \alpha'_{i_{m-1}}$. Recall that $\alpha_{i_m} = \alpha_{i_{m-1}}(1-\alpha_{i_{m-1}} q_{i_{m-1}} b_{i_{m-1}})$ and $\alpha'_{i_m} = \alpha'_{i_{m-1}}(1-\alpha'_{i_{m-1}} q'_{i_{m-1}} b_{i_m})$. As $\alpha_{i_{m-1}} q_{i_{m-1}} = \alpha'_{i_{m-1}} q'_{i_{m-1}}$, $\frac{\alpha_{i_m}}{\alpha'_{i_m}} = \frac{\alpha_{i_{m-1}}}{\alpha'_{i_{m-1}}}$. By the induction hypothesis, $\frac{\alpha_{i_{m-1}}}{\alpha'_{i_{m-1}}} < 1$ and hence $\alpha_{i_m} < \alpha'_{i_m}$.

Finally, using the fact that $\alpha'_{i_m} q'_{i_m} = \alpha_{i_m} q_{i_m}$, for all $m \geq 1$,

$$q'_i = q'_{i_m} < q_{i_m} = q_i,$$

concluding the proof of the Claim. □

As a final argument for Step 1, note that by Claim 5,

$$\sum_{i \in \Delta'} q'_i < \sum_{i \in \Delta'} q_i.$$

Furthermore, by Claim 2,

$$\sum_{i \in \Delta'} q_i < \sum_{i \in \Delta} q_i.$$

so that

$$\sum_{i \in \Delta'} q'_i < \sum_{i \in \Delta} q_i.$$

contradicting the fact that

$$\sum_{i \in \Delta'} q'_i = 1 = \sum_{i \in \Delta} q_i.$$

Step 2: For any target $i \in \Delta = \Delta'$, the sequence of preceding targets is the same in the attack paths p and p'.

Claim 6. *The set of first targets are identical, $\Delta_0 = \Delta'_0$.*

Proof. Let $i \in \Delta_0$ and suppose $i \in \Delta'_m$ with $m \geq 1$. If there is a path to i which does not intersect Δ', then the attacker has a profitable deviation by attacking i directly under E'. Hence all paths to i must intersect Δ'. But then because $\Delta = \Delta'$, the equilibrium path p to i must intersect Δ contradicting the fact that $i \in \Delta_0$. Hence $\Delta_0 \subseteq \Delta'_0$. Reverting the role of Δ_0 and Δ'_0, the same argument shows that $\Delta'_0 \subseteq \Delta_0$ □

Claim 7. *For any $i \in \Delta_m, m \geq 1$, the preceding target is the same on path p and on path p'.*

Proof. Suppose by contradiction that i has two different preceding targets, j and k on the paths p and p'. Because $\Delta = \Delta'$, the path from j to i does not intersect Δ' and the path from k to i does not intersect Δ. Hence, by Lemma 4, $b_k < b_j$ and $b_j < b_k$, a contradiction. □

Step 3: For any target $i \in \Delta = \Delta'$, the attack probabilities are the same, $q_i = q'_i$.

For a fixed set of targets Δ and an attack tree $\mathcal{T}$, letting U denote the attacker's equilibrium utility, we characterize equilibrium attack probabilities and defense investments as the solutions to the system of equations

$$\begin{aligned} b_i \prod_{k \preceq i} (1 - x_k) &= U, \\ x_i &= \prod_{k \prec i} (1 - x_k) q_i b_i, \\ \sum_i q_i &= 1 \end{aligned}$$

where the first equations capture the attacker's indifference over targets, the second equations the nodes' best response defense investments, and the last equation guarantees that probabilities belong to the simplex. The attacker's indifference conditions take a different form for first targets and subsequent targets.

$$\begin{aligned} x_i &= 1 - \frac{U}{b_i} \text{ if } i \in \Delta_0, \\ x_i &= 1 - \frac{b_k}{b_i} \text{ if } i \notin \Delta_0 \text{ with } k(i) \text{ the immediate predecessor of } i \end{aligned}$$

To compute equilibrium probabilities, we multiply each of the equations defining equilibrium defense investments by $(1 - x_i)$ t obtain

$$x_i(1 - x_i) = b_i \prod_{k \preceq i} (1 - x_k) q_i = U q_i.$$

and replacing x_i, we obtain

$$\begin{aligned} q_i &= \frac{1}{b_i}(1 - \frac{U}{b_i}) \text{ if } i \in \Delta_0, \\ q_i &= \frac{b_k}{b_i U}(1 - \frac{b_k}{b_i}) \text{ if } i \notin \Delta_0 \text{ with } k(i) \text{ the immediate predecessor of } i \end{aligned}$$

Let $F(U) \equiv \sum_i p_i(U)$. Since, for every i, $p_i(U)$ is a strictly decreasing function, $F(U)$ is strictly decreasing. Hence the equation $F(U) = 1$ has at most one solution. This concludes the proof of the theorem.

□

Proof of Proposition 2

Proof. In a line with first target b_i, the equilibrium distribution is given by

$$\begin{aligned} q_i &= \frac{1}{b_i}(1 - \frac{U}{b_i}) \\ q_j &= \frac{b_{j-1}}{b_j U}(1 - \frac{b_{j-1}}{b_j}) \text{ for } j > i \end{aligned}$$

Let

$$F(U) \equiv q_i(U) + \sum_{j>i} q_j(U) = \frac{1}{b_i}(1 - \frac{U}{b_i}) + \sum_{j>i} \frac{b_{j-1}}{b_j U}(1 - \frac{b_{j-1}}{b_j}).$$

We compute

$$F(b_i) = \sum_{j=i+1}^{m} (1 - \frac{b_{j-1}}{b_j}) \frac{b_{j-1}}{b_i b_j},$$

and

$$\begin{aligned} F(b_{i-1}) &= \frac{1}{b_i}(1 - \frac{b_{i-1}}{b_i}) + \sum_{j>i} \frac{b_{j-1}}{b_j b_{i-1}}(1 - \frac{b_{j-1}}{b_j}), \\ &= \sum_{j=i}^{m} (1 - \frac{b_{j-1}}{b_j}) \frac{b_{j-1}}{b_{i-1} b_j}. \end{aligned}$$

Given that b_i is the first target such that $\sum_{j=i+1}^{m}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$, we have

$$F(b_i) \leq 1 < F(b_{i-1}).$$

At an equilibrium, we must have $F(U) = 1$. The function $F(U)$ is strictly decreasing in U, and hence $b_{i-1} < U \leq b_i$. As $b_{i-1} > U$, for any target b_j in the increasing sequence with $j > i$, $b_j < U$ and hence $b_j \notin \Delta$. As $b_i \leq U$, for any $j \geq i$, $0 \leq q_j$, and hence every target $j \geq i$ is attacked with positive probability so that $b_j \in \Delta$, completing the proof of the Proposition. □

Proof of Proposition 3

Proof. Using equations (6), the defense investments of all targets in $\Delta \setminus \Delta_0$ except for node j are unaffected by the addition of the new link ij as the immediate predecessors of the targets remain the same as under the original network $\mathcal{G}$. Now consider how the addition of the link ij affects the function $F(U) = \sum_{i\in\Delta} q_i(U)$, by letting $F'(U) = \sum_{i\in\Delta} q'_i(U)$ denote the sum of probabilities after the change. Recall that both F and F' are strictly decreasing functions of U and let U'^* and U^* be the values for which $F'(U) = 1$ and $F(U) = 1$ respectively. We compute

$$F'(U) - F(U) = q'_j(U) - q_j(U) = \frac{(b_k - b_i)(b_k + b_i - b_j)}{b_j^2 U}.$$

As $b_k > b_i$, if $b_k + b_i - b_j > 0$, $F'(U) > F(U)$ for all U. But this implies that $U'^* > U^*$, the attacker's utility is higher after the addition of link ij. As U increases, we use equations (4) to deduce that the equilibrium investments of all first targets go down. Using equations (5) and (7), the equilibrium attack probabilities of all nodes but node j go down, and hence as $\sum_i q_i = 1$, the equilibrium attack probability of node j increases.

If $b_i + b_k < b_j$, $F'(U) < F(U)$ and similar arguments show that the equilibrium attacker's utility goes down, the defense investments of all first targets goes up, and the equilibrium probabilities of all nodes but node j go up. □

Proof of Proposition 4

Proof. All nodes $l \in \Delta \setminus \Delta_0$ keep the same immediate predecessor and hence the same equilibrium defense investment. Now let $F'(U)$ denote the sum of probabilities after the addition of the link $0i$. A straightforward computation shows that

$$F'(U) - F(U) = q'_j(U) - q_j(U) = (1 - \frac{U}{b_j})\frac{1}{b_j} - \frac{(b_j - b_k)b_k}{b_j^2 U} = \frac{1}{b_j^2 U}[U(b_j - U) - (b_j b_k)b_k].$$

As opposed to the case of Proposition 3, the sign of $F'(U) - F(U)$ depends on the value of U, and there is no necessary and sufficient condition on the target values which allows us to sign $F'(U) - F(U)$. Let $G(U) = U(b_j - U) - (b_j - b_k)b_k$. $F'(U) > F(U)$ if $G(U) > 0$ and $F'(U) < F(U)$ if $G(U) < 0$. Notice that G is a concave function with $G(0) = -(b_j - b_k)b_k < 0$ and $G(b_k) = 0$. Clearly, there is no sufficient condition guaranteeing that $G(U) > 0$ for all $U \in [0, b_k]$. A sufficient condition for $G(U) < 0$ for all $U \in [0, b_k]$ is $G'(b_k) > 0$, which is easily seen to be equivalent to $b_j > 2b_k$.

Hence if $b_j > 2b_k$, $F'(U) < F(U)$ for all $U \in [0, b_k)$. If $F'(U) < F(U)$, as in the proof of Proposition 3, the attacker's equilibrium utility goes down, the equilibrium investments of all first targets fo up, and the equilibrium probabilities of all nodes but node i go up, implying that the probability of attack of node i goes down. $\square$

Proof of Proposition 5

Proof. Consider the equations defining the equilibrium distribution over targets and equilibrium defense investments, (4 - 8). From equations (6), x_i goes up, x_l goes down and x_j remains constant for any other $j \notin \Delta_0$. Now let $F(U) = \sum_i q_i(U)$. Differentiation F with respect to b_i and using equations (7) ,

$$\frac{\partial F}{\partial b_i} = \frac{1}{U}[\frac{b_{k(i)}(2b_{k(i)} - b_i)}{b_i^3} + \sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2}].$$

Now,

$$\frac{dF}{db_i} = \frac{\partial F}{\partial b_i} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial b_i} = 0,$$

so that

$$\frac{\partial U}{\partial b_i} = -\frac{\frac{\partial F}{\partial b_i}}{\frac{\partial F}{\partial U}}.$$

Hence, if $\frac{\partial F}{\partial b_i} > 0$, then $\frac{\partial U}{\partial b_i} > 0$. Next, we use equations (4) to deduce that x_j decreases for all $j \in \Delta_0$ and equations (5) to conclude that q_j decreases for all $j \in \Delta_0$ and (7) to show that q_j decreases for all $j \notin \Delta_0$ which is not equal to i nor an immediate successor of i. While we conclude that the sum of probabilities $q_i + \sum_{l \succ i} q_l$ must go up, we cannot deduce whether p_i goes up or not.

A similar reasoning gives the opposite result in the case where $\frac{\partial F}{\partial b_i} < 0$. $\square$

Proof of Proposition 6

Proof. As in the proof of Proposition 5, a change in b_i does not affect the defense investments of nodes in Δ which are not immediate successors of i by equations (6). Next, using equations (5) and (7), we compute

$$\frac{\partial F}{\partial b_i} \equiv G(U) = \frac{2U - b_i}{b_i^2} + \frac{1}{U}\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2}.$$

Observe that, as opposed to the case of Proposition 5, the sign of $\frac{\partial F}{\partial b_i}$ depends on the value of U. Hence there is no necessary and sufficient condition on the target values under which the sign of $\frac{\partial F}{\partial b_i}$ can be established, and we look instead for sufficient conditions.

Consider first the case where $\sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2} > 0$. Then the function $G(\cdot)$ is a convex function of U, with $\lim_{U\to 0} G(U) = +\infty, G(b_i) > 0$ and $\frac{\partial G}{\partial U} = \frac{2}{b_i^2} - \sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2 U^2}$. Hence $G'(U) < 0$ whenever $U < U^* = b_i\sqrt{\frac{\sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2}}{2}}$ and $G'(0) > 0$ whenever $U > U^*$. A sufficient condition for $G(U) > 0$ is thus that $G(U^*) > 0$. We verify that $G(U^*) > 0$ if and only if $\sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2} > \frac{1}{8}$.

Next suppose that $\sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2} < 0$. Then the function $G(\cdot)$ is an increasing function of U with $\lim_{U\to 0} G(U) = -\infty$. In addition, as $U \le \min_{j\in\Delta} b_j$, $U \le b_i$. Hence a sufficient condition for $G(U) < 0$ is $G(b_i) < 0$. We verify that $G(b_i) > 0$ if and only if $\sum_{l,l\succ i} \frac{b_l - 2b_i}{b_l^2} < -1$.

Once the sign of $\frac{\partial F}{\partial b_i}$ is established, we follow the same steps as in the proof of Proposition 5 to compute the sign of $\frac{\partial U}{\partial b_i}$ and the comparative statics effects of an increase in b_i on the equilibrium investment levels of first targets different from i and equilibrium attack probabilities for all nodes but i and its immediate successors. □

Proof of Theorem 3

Proof. Suppose not, and let the attacker use a mixed strategy q with a support of at least two nodes.

First, we show that there is no branch in the tree containing two or more nodes in the support of q. Suppose this is false, and there is indeed one such branch. Without loss of generality, let $\{1, \ldots, I-1, I\}$ be the set of targets on this branch.

Consider any node numbered k in the support. If node $k+1$ observes the package, it updates the probability she is the target to

$$q'_{k+1} = \frac{q_{k+1}}{q_{k+1} + q_{k+2} + ... + q_I}.$$

Then, because her payoff is given by

$$-q'_{k+1} b_{k+1} - \frac{1}{2} x_{k+1}^2$$

node $k+1$ will intercept with probability $x_{k+1} = q'_{k+1} b_{k+1}$.

Consider node I. If the package is observed to have passed $I-1$, I believes with probability 1 that she is the target and will therefore intercept with $x_l = b_l$.

The ex ante expected payoff to the sender from targeting I is thus

$$b_I(1 - b_I)...(1 - b_k q'_k)..(1 - b_1 q'_1)$$

For $I-1$, the analogous expected payoff is

$$b_{I-1}(1 - q'_{I-1} b_{I-1})...(1 - b_k q'_k)..(1 - b_1 q'_1)$$

The two preceding equations give us

$$b_I(1 - b_I) = b_{I-1}$$

since expected payoffs have to be equal for all points in the support. However, this equation violates the first statement of Assumption 2. Therefore, I and $I-1$ cannot both be in the support.

Second, suppose the support of q has a non-negative intersection with the sets of nodes on *two different* branches of the tree. Let i and j be the last nodes to be attacked on these branches. In view of the preceding argument, neither i nor j can have predecessors that are attacked with positive probability. If the object is detected at i, then i is the target with probability one. The expected payoff to the attacker from sending the package to i is $b_i(1-b_i)$. For similar reasons, the expected payoff from sending the object to j is $b_j(1-b_j)$. For q to be an equilibrium. we need

$$b_i(1-b_i) = b_j(1-b_j)$$

This violates the second statement of Assumption 2.

□

Proof of Lemma 5

Proof. We prove it by induction on K. The Lemma is clearly true if $K = 2$. Suppose it is true for all increasing sequences up to length $K-1$.

Let b_i satisfy equations (9) and (10) for the truncated sequence $(b_1, b_2, \ldots, b_{K-1})$. If $b_i \geq \Pi_{k=1}^{K-i}(1-b_{i+k})b_K$, there is nothing to prove. Suppose, then that

$$\Pi_{k=1}^{K-i}(1-b_{i+k})b_K > b_i \tag{13}$$

From the induction hypothesis,

$$b_i \geq \Pi_{k=1}^{K-1-i}(1-b_{i+k})b_{K-1} \tag{14}$$

The last two equations give

$$(1-b_K)b_K \geq b_j \forall j \neq K. \tag{15}$$

it follows that equation (9) is satisfied. This completes the proof of the Lemma.

□

Proof of Theorem 4

Proof. Sufficiency. Let $i \in B = \{1, \ldots K\}$ be a target that satisfies equations (9) and (10) on B and (11).

In order to show that i can be supported as a sequential equilibrium target, we need to construct one set of out-of-equilbrium beliefs that will rule out any profitable deviation for the attacker.

Consider out-of-equilibrium beliefs such that

$$\begin{aligned}
\beta_{i+1} &= 1 - \sum_{j \in N-P(i,B), j \neq i} \beta_j \\
\beta_{i+k} &= \epsilon^{k-1} \text{ for } k = 2, \ldots, K-i \\
\beta_j &= \epsilon^{|P(j,B')|+1} \text{ for all } j \in B' \neq B
\end{aligned}$$

Let the attacker choose to attack i. Then, if any $j \in P(i, B)$ observes that the object has reached its own location, it will believe that $i$is the target and will check with probability 0. Suppose $j \notin P(i, B)$ and $j \neq i$. Then, for any $B' \in \mathcal{B}$, and $j \in B'$,

$$q_j = \frac{\beta_j}{\sum_{k \in S(j,B')} \beta_k + \beta_j}$$

Notice that

$$\lim_{\epsilon \to 0} q_j = 1 \; \forall j \notin (P(i, B) \cup \{i\} \tag{16}$$

To show that the attacker's strategy is optimal, we check for deviations.

Suppose $j \in P(i, B)$. If the sender deviates and targets j, j does not check since j is on the equilibrium path. However, equation (9) ensures that the sender does not profit from the deviation.

Suppose $j \notin P(i, B)$. Then, use equations (10), (11) and (16) to conclude that the sender does not profit from the deviation. Hence there is a sequential equilibrium where node i is attacked.

Necessity. Consider first a node $i \in B$ such that equation (9) fails and let $j \in P(i, B)$ be such that $b_j > b_i(1 - b_i)$. If there is a sequential equilibrium where i is attacked, node j does not choose any defense, and hence the attacker has an incentive to deviate and attack node j, contradicting the fact that i is attacked in equilibrium.

Suppose next that equation (10) fails for node i and let $j \in S(i, B)$ be such that $\Pi_{k \in S(i,B) \cap P(j,B)}(1 - b_k)b_j > b_i$. Notice that for all $k \in S(i, B) \cap P(j, B)$, $x_k \leq b_k$. Hence the probability that the package arrives at node j is bounded above by $(1 - b_i)\Pi_{k=1}^{j-i}(1 - b_{i+k})$. So by attacking node j, the attacker obtains at least

$$b_j(1 - b_i)\Pi_{k=1}^{j-i}(1 - b_{i+k}) > b_i(1 - b_i),$$

showing that this deviation is profitable, and contradicting the fact that i is attacked in equilibrium.

Finally, suppose that equation (11) fails for node $i \in B$. Choose any $B' \in \mathcal{B}$ distinct from B and $j \in B'$. For any predecessor k of j (if any), $x_k \leq b_k$ and $x_j \leq b_j$. Hence, violation of equation (10) will again imply that the attacker can profitably deviate from attacking i. □

Proof of Lemma 6

Proof. Suppose the Lemma is false and for some $i < n$, $y_{i+1} = y_i + \epsilon$.

Suppose, first that $r_i > 0$. Then, consider $y'_i = y_{i+1}$ and $y'_{i+1} = y_i$. Then, $\beta'_i < \beta_i$ and $\beta'_{i+1} = \beta_{i+1}$, while cost remains the same. Clearly, C gains.

If $r_i = 0$, then equalize defense outlay on nodes i and $i+1$, so that $y'_i = y'_{i=1} = y_i + \frac{\epsilon}{2}$. it is easy to check that C gains agains. □

Proof of Proposition 7

Proof. We first show that all equilibria must have the same support on the line. Consider two equilibria with supports Δ and Δ'.

Let m be such that $b_m > b_i$ for all $i \neq m$. It is clear that $m \in \Delta \cap \Delta'$. Moreover, no node $(m+k) \in \Delta$ since $\beta_m b_m > \beta_m b_{m+k}$. Similarly, $(m+k) \notin \Delta'$. Hence, both supports have the same final target m.

Now suppose that the equilibria have two different supports $\Delta \neq \Delta'$. As the two equilibria have the same final target, there exists a target l such that all targets $k \geq l$ are in both supports Δ and Δ', and there exists a target i preceding l in Δ which does not belong to Δ' with no target between i and l in Δ'. We distinguish between two cases: (i) there exists a target preceding l in Δ' and (ii) there is no target preceding l in Δ'.

Case 1. There exists a target preceding l in Δ' Let j be the target preceding i in Δ'. By construction $j < i$. Because there is no target between j and l in Δ', $b_j = b_l(1-y'_l)^{l-j}b_l$. We also note that $b_j \leq \prod_{j<k\leq l}(1-y_k)b_l$. Furthermore, because $y_k \geq y_l$ for all $k < l$, and there exists a node i attacked between k and l, $\prod_{j<k\leq l}(1-y_k)b_l < (1-y_l)^{l-j}b_l$. Hence

$$(1-y'_l)^{l-j} = b_j < (1-y_l)^{l-j},$$

so that $y'_l > y_l$. Finally, because $i \in \Delta$ but $i \notin \Delta'$,

$$b_i = (1-y_l)^{l-i}b_l \leq (1-y'_l)^{l-i}b_l,$$

so that $y'_l \leq y_l$, a contradiction.

Case 2. There is no target preceding l in Δ'. We then compute y'_l as the solution to the equation:

$$y'_l = (1-y'_l)^{l-1}b_l.$$

Similarly, letting j denote the first target attacked in Δ, we have

$$y_j = (1-y_j)^{j-1}b_j = (1-y_j)^{j-1}\prod_{j<k\leq l}(1-y_k).$$

But as $y_j \geq y_k$ for all $k > j$ with one strict inequality because there is at least one target before l in Δ,

$$(1-y_j)^{l-1} < y_j < (1-y_l)^{l-1}b_l.$$

As the function $g(y) = y - (1-y)^{l-1}b_l$ is increasing, we conclude that $y_j > y'_l$ so that

$$(1-y_l)^{l-1}b_l > y_j > y'_l = (1-y'_l)^{l-1}b_l,$$

showing that $y'_l > y_l$. But as $i \in \Delta$ but $i \notin \Delta'$, $y'_l \leq y_l$, a contradiction.

Finally, we show that for a fixed support Δ, there is a unique equilibrium attack distribution. Let $i = 1$ be the first target. For any two consecutive targets $i-1$ and i, let $d(i)$ denote the length of the path between i and $i-1$. The equilibrium attack probabilities and defense investments can be computed as the solutions to the system of equations:

$$
\begin{aligned}
y_1 &= b_1(1-y_1)^{d(1)-1} && (17)\\
b_i(1-y_i)^{d(i)} &= b_{i-1}, \text{ for } i \in \Delta, i \neq 1 && (18)\\
y_i &= b_i \sum_{j \geq i} r_j \prod_{k \neq i}(1-y_k) \text{ for } i \in \Delta, i \neq 1 && (19)\\
\sum_i r_i &= 1. && (20)
\end{aligned}
$$

Notice that equations (17) and (18) uniquely determine the investment values y_i for all $i \in \Delta$. Given the defense investments, equations (19) and (20) uniquely determine the equilibrium attack probabilities r_i for all $i \in \Delta$, completing the proof of the Proposition. □

Proof of Proposition 8

Proof. (i) Since $\mathcal{M}_i = -U_i$, it follows that

$$
\mathcal{M}_i = x_i - \frac{x_i^2}{2}
$$

We can write

$$
\begin{aligned}
y_i &= \prod_{k<i}(1-y_k) \sum_{j \in \Delta, j \geq i} \prod_{i<l\leq j}(1-y_l) b_j r_j\\
&= \alpha_i q_i b_i + \alpha_i \sum_{j \in \Delta, j>i} \beta_j r_j b_j \text{ for } i < n\\
&= \alpha_i q_i b_i \text{ for } i = n
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathcal{L}_i &= \alpha_i q_i b_i (1-x_i) + \frac{x_i^2}{2}\\
&= x_i(1-x_i) + \frac{x_i^2}{2} - \alpha_i \sum_{j \in \Delta, j>i} \beta_j r_j b_j\\
&< x_i - \frac{x_i^2}{2} \text{ for } i < n\\
&= x_i - \frac{x_i^2}{2} \text{ for } i = n
\end{aligned}
$$

Using $x_i = y_i = 1 - \frac{b_{i-1}}{b_i}$ for $i > 1$, we get $\mathcal{M}_i > \mathcal{L}_i$ for $i = 2, 3 \ldots, n-1$ and $\mathcal{M}_n = \mathcal{L}_n$.

(ii) Rewrite $\mathcal{L}$ and $\mathcal{M}$ as follows

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{n} \beta_i^c r_i b_i + \sum_{i=1}^{n} \frac{y_i^2}{2}\\
\mathcal{M} &= \sum_{i=1}^{n} \beta_i q_i b_i + \sum_{i=1}^{n} \frac{x_i^2}{2}
\end{aligned}
$$

From the attacker's equilibrium condition,

$$\beta_i^c b_i = \beta_j^c b_j \text{ and } \beta_i b_i = \beta_j b_j \text{ for all } i, j$$

$$\begin{aligned}
\mathcal{M} - \mathcal{L} &= \beta_1 b_1 - \beta_1^c b_1 + \frac{x_1^2}{2} - \frac{y_1^2}{2} \\
&= (1 - q_1 b_1) b_1 + \frac{(q_1 b_1)^2}{2} - (1 - b_1) b_1 - \frac{(b_1)^2}{2} \\
&= \frac{(q_1 b_1)^2}{2} - q_1 b_1^2 + \frac{(b_1)^2}{2} \\
&= \frac{(b_1)^2}{2} (1 + q_1^2 - 2q_1) \\
&> 0 \text{ for all } q_1 \in (0, 1)
\end{aligned}$$

(iii) This follows straightaway since $\beta^c b_1 = (1 - b_1) b_1 < (1 - q_1 b_1) b_1 = \beta b_1$. $\square$

Appendix B: Characterization of the support in the cooperative model

Here, we characterize the support of A's equilibrium attack strategy in the cooperative model, as a function of the target values. Define ξ_i to be the unique solution in $[0,1]$ to

$$b_i(1-\xi_i)^{i-1} = \xi_i.$$

Let i^* be the node in the line for which ξ_i is maximal if $b_1 \leq \frac{1}{2}$. If $b_1 > \frac{1}{2}$, let $i^* = 1$. Notice that whenever $i \geq 2, \xi_i < \frac{1}{2}$ and if $i = 1$ and $b_i < \frac{1}{2}$ then $\xi_1 < \frac{1}{2}$. Furthermore, if $\xi_i < \frac{(}{1})2$, then $\xi_i < \xi_j \Leftrightarrow \xi_i(1-\xi_i) < \xi_j(1-\xi_j)$.

Next, for any i, compute, for any $j > i$,

$$\zeta_j(i) = 1 - (\frac{b_i}{b_j})^{\frac{1}{j-i}}.$$

and let $j^*(i)$ be the node with the highest value $\zeta_j(i)$.

Proposition 9. *For any sequence $b_1,..,b_n$, compute the sequence $i_1^* = i^*, i_2^* = j^*(i^*), ..., i_m^* = j^{*(m-1)}(i^*)$, with $b_{i_k^*} > b_j$ for all $j > i_m^*$. The equilibrium support is given by $\Delta = \{i_1^*, i_2^*, ...i_m^*\}$.*

Proof. We first prove that i^* is the first target attacked in the support.

Suppose that the first node attacked satisfies $i < i^*$. Then $i^* \geq 2$ so that $b_i < \frac{1}{2}$ and hence $\xi_i < frac12$ for all i and $\xi_{i^*} > \xi_i$ for all $i \neq i^*$. Let $i_1, i_2, ..., i^m$ be the nodes attacked before i^*. We have $y_{i^1} = \xi_{i^1} < \xi_{i^*}$ and, by Lemma 6 $x_{i^j} < x_{i^1}$ for all $j = 2, ..., m+1$. Hence

$$b_{i^*}(1-y_{i^1})^{i^1}(1-y_{i^2})^{i^2-i^1}...(1-y_{i^{m+1}})^{i^*-i^m} > b_{i^*}(1-\xi_i^*)^{i^*}.$$

Now, as $\xi_i < \frac{1}{2}$ for all i,

$$b_{i^*}(1-x_i^*)^{i^*} = \xi_{i^*}(1-\xi_{i^*}) > \xi_{i^1}(1-\xi_{i^1}) = b_{i^1}(1-\xi_{i^1})^{i^1}.$$

Hence

$$b_{i^*}(1-y_{i^1})^{i^1}(1-y_{i^2})^{i^2-i^1}...(1-y_{i^m})^{i^*-i^m} > b_{i^1}(1-y_{i^1}),$$

showing that the attacker has an incentive to wait, skip all the nodes $i_1, .., i^m$ and attack b_{i^*} as his first target. This contradiction shows that the attacker cannot attack any node $i < i^*$ as a first target.

Suppose now that the first node attacked satisfies $i > i^*$. If $b_1 > \frac{1}{2}$, then by attacking 1, the attacker obtains $b_1(1-y_i) = b_1(1-\xi_i)$. As $b_1 > \frac{1}{2} > \xi_i$,

$$b_1(1-\xi_i) > \xi_i(1-\xi_i) = b_i(1-\xi_i)^i,$$

so that the attacker has an incentive to deviate and attack node 1. If $b_1 < \frac{1}{2}$, then by attacking i^*, the attacker obtains $b_{i^*}(1-y_i)^{i^*} = b_{i^*}(1-\xi_i)^{i^*}$. As $\xi_{i^*} > \xi_i$,

$$b_{i^*}(1-\xi_i)^{i^*} > b_{i^*}(1-\xi_{i^*})^{i^*} = \xi_{i^*}(1-\xi_{i^*}) > \xi_i(1-\xi_i) = b_i(1-\xi_i)^i,$$

so that the attacker has an incentive to deviate and attack node i^*. This argument shows that the attacker cannot attack $i > i^*$ as a first target, completing the proof that i^* is the first target in the support.

Next, we prove that if i is attacked, the next target attacked must be $j^*(i)$.

Suppose that the next target is $j < j^*(i)$. Let $j_1, .., j_m$ be the nodes attacked between i and $j^*(i)$. We have $y_{j^1} = \zeta_j(i^1) < \zeta_{j^*(i)}$ and by Lemma 6, $y_{j^k} < y_{j^1}$ for $k = 2, ..., m$. Hence

$$b_{j^*(i)}(1-y_{j^1})^{j^1-i}(1-y_{j^2})^{j^2-j^1}...(1-y_{j^{m+1}})^{j^*(1)-j^m} > b_{j^*(i)}(1-\zeta_{j^*(i)}(i))^{j^*(i)-i}.$$

Now, by the definition of $j^*(i)$,

$$b_{j^*(i)}(1-\zeta_{j^*(i)}(i))^{j^*(i)-i} = b_j(1-\zeta_j(i))^{j-i} = b_i \text{ for } all j > i,$$

so that

$$b_{j^*(i)}(1-y_{j^1})^{j^1-i}(1-y_{j^2})^{j^2-j^1}...(1-y_{j^{m+1}})^{j^*(1)-j^m} > b_{j^*(i)}(1-\zeta_{j^*(i)}(i))^{j^*(i)-i} = b_i,$$

so that the attacker has an incentive to skip all nodes $j^1, ..., j^m$ and attack $j^*(i)$, a contradiction which shows that the attacker cannot attack any node $j < j^*(i)$ after attacking node i.

Finally, suppose that the next target is $j > j^*(i)$. Again, because $\zeta_{j^*(i)} > \zeta_j$,

$$b_{j^*(i)}(1-\zeta_j)^{j^*(i)-i} > b_{j^*(i)}(1-\zeta_{j^*(i)}(i))^{j^*(i)-i} = b_i,$$

so that the attacker has an incentive to attack $j^*(i)$. This shows that the attacker must attack $j^*(i)$ after attacking i, completing the proof of the Proposition. $\square$